A FINITE GRAPH IS HOMEOMORPHIC TO THE REEB GRAPH OF A MORSE–BOTT FUNCTION

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Abstract. We prove that a finite graph (allowing loops and multiple edges) is homeomorphic (isomorphic up to vertices of degree two) to the Reeb graph of a Morse–Bott function on a smooth closed n-manifold, for any dimension $n \geq 2$. The manifold can be chosen orientable or non-orientable; we estimate the co-rank of its fundamental group (or the genus in the case of surfaces) from below in terms of the cycle rank of the graph. The function can be chosen with any number $k \geq 3$ of critical values, and in a few special cases with $k < 3$. In the case of surfaces, the function can be chosen, except for a few special cases, as the height function associated with an immersion in $\mathbb{R}^3$.

1. Introduction

Given a smooth manifold M, the *Reeb graph* R_f of a smooth function $f : M \to \mathbb{R}$ is a topological space obtained by contracting the connected components of the level sets of f to points, endowed with the quotient topology. The Reeb graph shows the evolution of the topology of the level sets, thus providing important information on the behavior of the function. The notion of the Reeb graph has found important applications in computer graphics, data analysis and visualization, geometric model databases, shape analysis, and other areas of applied mathematics and computer science [1].

The notion of the Reeb graph was introduced in 1946 for Morse functions on a compact manifold [17]. Later, its topological properties were studied for more general types of functions on a compact manifold., e.g., functions with finite number of critical points [9, 15, 19], functions with finite number of critical values [13], Morse–Bott functions [12], smooth functions [6], and even arbitrary continuous functions on a topological space [7].

The problem of whether a finite graph can be realized as the Reeb graph of some function was first studied in 2006 by Sharko [19]. Since a finite graph can be considered as a one-dimensional CW complex, the realization problem can be studied in two ways: up to combinatorial isomorphism of graphs or up to homeomorphism of complexes (i.e., isomorphism of graphs up to vertices of degree two).

As to combinatorial isomorphism, not every graph can be the Reeb graph of some function: for example, a Reeb graph cannot have loops (edges incident to one vertex). If we consider the problem in a specific class of functions, it imposes some additional restrictions on the type of the graph. For instance, the Reeb graph of a Morse function must have at least two vertices of degree 1. For smooth functions with a finite number of critical points, Sharko [19, Theorem 2.1] proved a criterion for the isomorphic realization in terms of graphs admitting so-called good orientation. In a class of Morse–Bott functions, conditions for isomorphic realizability of a finite graph as the Reeb graph are even more complicated [12, Theorem 1.2].

Previous results on realization of a finite graph as the Reeb graph of a function of some class up to homeomorphism also impose some restrictions on the graph. Masumoto and Saeki [13,

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Theorem 2.1] proved that a finite graph without loops is homeomorphic to the Reeb graph of a smooth function with a finite number of critical values on a closed surface (in fact, the graphs are isomorphic [13, Remark 2.3]). Michalak proved that a finite graph admitting good orientation is homeomorphic to the Reeb graph of a Morse function on a closed surface [15, Theorem 5.4, Remark 5.5] and on a closed higher-dimensional manifold [14, Theorem 6.4].

Morse–Bott functions are an important generalization of Morse functions: unlike Morse functions, they reflect the symmetries often observed both in formulas typically used in practical calculations and in real-world phenomena and artifacts. Thus these functions are more suitable for use in applied mathematics and in many areas of pure mathematics.

In this paper, we show that realization of a finite graph up to homeomorphism as the Reeb graph of a Morse–Bott function does not impose any restrictions on the graph: Any finite graph G (allowing multiple edges and loops) is homeomorphic to the Reeb graph of a Morse–Bott function f on a smooth closed n-dimensional manifold M, for any dimension $n \geq 2$ (Theorem 3.4). Note that, as we have mentioned above, in the case of *isomorphism*, as well as in the case of a *Morse* function, rather strong restrictions are imposed on the graph.

We further study what manifolds M and what functions f can be chosen in this construction for a given graph G. As to the manifolds, we show that M can be chosen as a two-dimensional surface. Specifically, the desired function exists on a given surface M, orientable or non-orientable, if and only if for the genus $g(M)$ it holds (Theorem 5.4)

$$g(M) \geq \begin{cases} b_1(G) & \text{if } M \text{ is orientable,} \\ 2b_1(G) & \text{if } M \text{ is non-orientable,} \end{cases}$$

or, in terms of in the co-rank of the fundamental group (Theorem 5.6),

$$\operatorname{corank}(\pi_1(M)) \geq b_1(G), \tag{1.1}$$

where $b_1(G)$ is the Betti number, called also the cycle rank, of the graph G.

As to the function f, we show that the function on the surface can be chosen with any number $k \geq 3$ of critical values, and in six special cases also with $k < 3$; in addition, in almost all cases the function f can be chosen as the height function associated with a suitable immersion of the surface M in $\mathbb{R}^3$ (Theorem 5.5).

These properties of M and f are, in most cases, independent, with exceptions related to functions with $k < 3$ critical values. Specifically, Theorems 5.5 summarizes the above observations in a criterion for a given finite graph G (with possible multiple edges and loops) to be homeomorphic to the Reeb graph of a Morse–Bott function f, with a given number k of critical values, on a smooth closed surface M of a given genus g, orientable or non-orientable. It also gives a criterion for the possibility to choose f as the height function associated with an immersion of M in $\mathbb{R}^3$.

These observations can be partly generalized to higher dimensions in rather obvious way, though we do not deep into details. In particular, we generalize Theorem 5.6, see (1.1), to higher dimensions only up to existence: For any $n \geq 2$, there exists an n-manifold M with $\operatorname{corank}(\pi_1(M)) = c$ allowing a desired function if and only if $c \geq b_1(G)$; the manifold can be chosen orientable or non-orientable (Theorem 5.7). However, unlike for surfaces, for higher dimension the co-rank of the fundamental group and orientability do not uniquely define the manifold. Currently we work on a generalization of Theorem 5.6 to an arbitrary dimension in a more complete way: Given a specific n-manifold M, $n \geq 3$, in our future work we will study whether a given graph G can be realized as the Reeb graph of a Morse–Bott function on M.

The paper is organized as follows. In Section 2, we give necessary definitions and basic facts. In Section 3, we prove the theorem on realization of a finite graph (up to homeomorphism) as the Reeb graph of a Morse–Bott function. In Section 4, we give some additional definitions and facts needed for the next section. Finally, in Section 5, we study the properties of the manifold (mainly surface) and of the function that can realize a given graph as the corresponding Reeb graph.

2. Definitions and useful facts

2.1. **Graphs.** Since there is no consensus in terminology related to graphs, we define here the terms as we will use them.

We consider finite *graphs* that allow multiple edges (also called *parallel edges* or a *multi-edge*) and loops (also called a *self-loop* or a *buckle*); a *loop* is an edge incident to only one vertex (note that in computational geometry, this term is used differently: there, a loop is an undirected cycle). A graph needs not to be connected. Two vertices are *adjacent* if they are joined by an edge. The *degree* $\deg v$ of a vertex v is the number of edges incident to it (loops are counted twice); a vertex v is an *isolated vertex* if $\deg v = 0$ and a *leaf* if $\deg v = 1$.

A graph is called *trivial* if it has only one vertex and no edges (a graph must have at least one vertex). A complete graph with n vertices is denoted by K_n; K_2 has one edge and geometrically represents a 1-simplex or a closed interval.

A *path graph* P_n is a tree with n vertices with exactly two leaves (all other vertices, if any, are of degree 2; note that we do not consider the trivial graph to be a path graph). A *cycle graph* C_n is a connected graph with all its n vertices being of degree 2 (again, we do not consider the trivial graph to be a cycle graph). *Smoothing* a graph is repeatedly smoothing out all vertices of degree 2 (removing the vertex and joining the two vertices adjacent to it by an edge; this is also called *lifting*) except those incident to a loop; smoothing does not change the graph up to homeomorphism. For example, smoothing P_n, $n \geq 2$, results in K_2. Smoothing C_n, $n \geq 1$, results in C_1; smoothing any other graph results in no vertices of degree 2 (though new loops may appear).

A graph can be considered as a one-dimensional CW complex. Two graphs are *homeomorphic* if they are homeomorphic as CW complexes. Equivalently, two graphs are homeomorphic when they become isomorphic after smoothing.

The *cycle rank* $b_1(G)$ of a graph G is the first Betti number of the graph considered as a one-dimensional CW complex; in computational geometry this value is called the *number of loops*.

A vertex v is called *cut vertex* (*articulation point*) if the number of connected components of $G \setminus v$ is greater than that of G. An (undirected) graph is *biconnected* if it is connected and has no cut vertices (not the same as *2-connected*). A *block* (also called *biconnected component* or, by some authors, *2-connected component*) of an (undirected) graph is a maximal biconnected subgraph. The blocks of a graph are edge disjoint and are attached to each other at shared vertices, which are cut vertices of the whole graph.

An *orientation* of an (undirected) graph G is an assignment of a direction to each of its edges; a graph with an orientation is called a *digraph*. The *indegree* $\deg_{in} v$ of a vertex v in a digraph is the number of edges incoming to v and the *outdegree* $\deg_{out} v$ of a vertex v is the number of edges outgoing from v; obviously, $\deg v = \deg_{in} v + \deg_{out} v$. A vertex v is a *source* if $\deg_{in} v = 0$, a *sink* if $\deg_{out} v = 0$, and an *internal vertex* if it is neither a source nor a sink. Obviously, a leaf is either a source or a sink.

By a *cycle* in a digraph, we mean a directed circuit that passes by each its vertex only once. A digraph is *acyclic* if it has no directed cycles. A *block* in a digraph is a (directed) sub-graph that is a block in the corresponding undirected graph.

Let s, t be vertices of G. A *bipolar orientation*, or *st-orientation*, of G is an orientation that turns G into a directed acyclic graph with s being its only source and t its only sink. For an edge e of G, an *e-bipolar* orientation of G is its *st*-orientation, where s and t are vertices incident to e; if the orientation on e is given, we assume s to be the tail and t the head of e.

Theorem 2.1 ([4, Theorem 4.1]). *Let e be an edge of an (undirected) graph G without loops. Then G admits an e-bipolar orientation if and only if G is biconnected.*

2.2. **Morse–Bott functions, Reeb graph, and the height function.** We will consider only smooth closed manifolds (smooth compact manifolds without boundary) and smooth functions.

A *Morse–Bott function* is a smooth function $f: M \to \mathbb{R}$ on a manifold M, with its critical set being a closed submanifold and with the Hessian being non-degenerate in the normal direction. Connected components of its critical set are non-degenerate submanifolds. A Morse function is the special case where the critical manifolds are zero-dimensional. A constant function $f: M \to \text{const}$ on a closed manifold M is considered a Morse–Bott function; indeed, its critical set is a closed submanifold, namely, the whole M.

The *Reeb graph* R_f of a continuous function $f: X \to \mathbb{R}$ on a topological space X is the quotient space $X/{\sim}$ endowed with the quotient topology, where the equivalence relation $x \sim y$ holds whenever x and y belong to the same *contour* (connected component of a level set) of f. The Reeb graph of a Morse–Bott function is a finite topological graph (one-dimensional CW complex) [3, 18].

If f is a differentiable function, then usually all its critical points are considered vertices of R_f, making the notion of the Reeb graph slightly different from the use of the term in the general case: for the differential function, its (combinatorial) Reeb graph can have vertices of degree 2, which do not make sense in the general (topological) definition of the Reeb graph. Consequently, one can consider the Reeb graphs of differentiable functions up to isomorphism (suitable for the special definition of the term) or up to homeomorphism (consistent with the general definition). In this paper, we will deal with the latter case: we consider the general definition of the notion of the Reeb graph, and thus we consider graphs up to homeomorphism.

The *height function* associated with an immersion $i: M \to \mathbb{R}^n$ of a manifold M in $\mathbb{R}^n$ with the coordinates $(x_1, \dots, x_n)$ is the composition $h = P \circ i: M \to \mathbb{R}$, where $P: \mathbb{R}^n \to \mathbb{R}$ is the projection defined by $P(x_1, \dots, x_n) = x_n$. This is a smooth function. There exist smooth functions, even with finitely many critical points, on a surface that cannot be realized as a height function [10, Theorem 1, Proposition 1]; see also Proposition 4.1.

3. A finite graph is homeomorphic to the Reeb graph of a Morse–Bott function

In this section, given a graph G, we will construct an embedding of a surface in $\mathbb{R}^3$ such that the associated height function realizes G as its Reeb graph up to homeomorphism. The process, each step of which is explained in detail below, is outlined in Figure 1. Next, we will generalize the result to arbitrary dimension. In the next sections, we will study the properties of the obtained construction.

We will need some auxiliary statements.

Internal vertices of a directed graph have degree greater than one. A finite graph admits an acyclic orientation such that all such vertices are internal:

Proposition 3.1. *Let G be a finite graph (allowing multiple edges and loops) without isolated vertices. Then there exists an orientation of G such that a vertex v is internal if and only if* $\deg v \neq 1$.

Proof. Consider blocks of G. If a block B_i is a loop, then it has a unique vertex, which is internal. In each non-loop block $B_i \neq K_2$ (complete graph of two vertices) of G, choose an edge e_i, choose an e_i-bipolar orientation in B_i (Theorem 2.1), and then reverse the orientation of e_i. Note that all vertices of B_i are internal.

Let now T be the subgraph of G which is the union of the K_2-blocks of G. Denote connected components of T by T_j. Each T_j is a tree, with all T_j being vertex disjoint, as well as edge disjoint

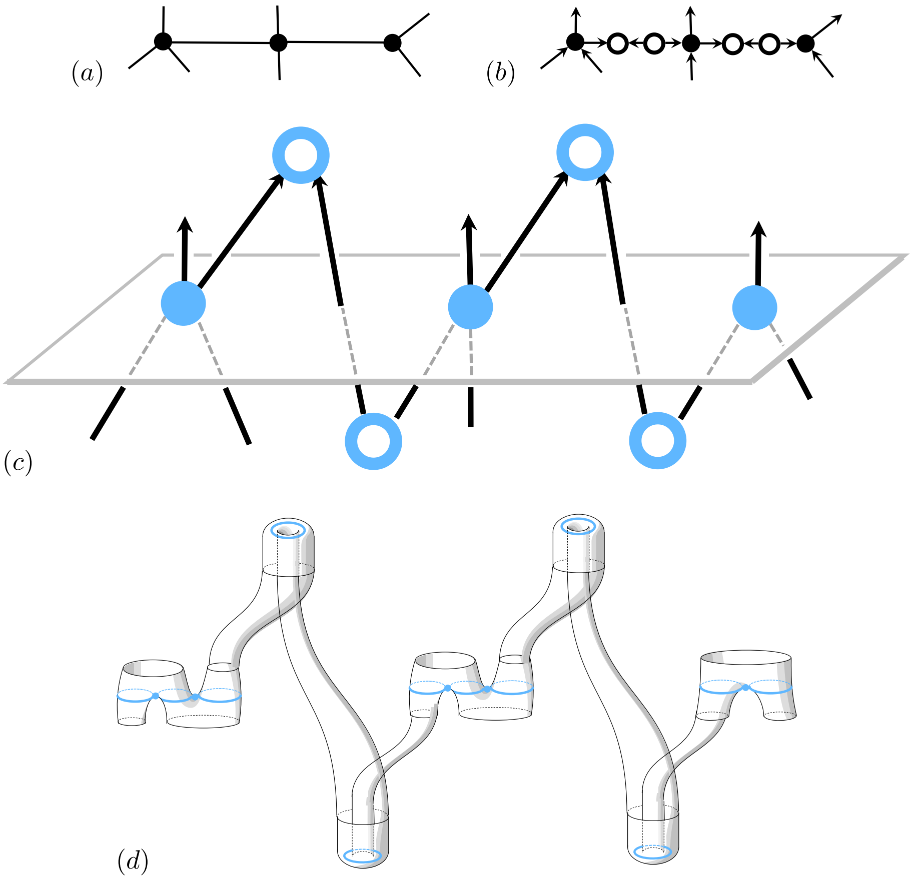

FIGURE 1. The process of building the desired immersion. (a) A fragment of the initial graph G. The loose ends of the edges are connected to vertices not shown in the picture. (b) Each edge is subdivided by two new vertices (white), and orientation is introduced such that vertices v with $\deg v \geq 3$ are intermediate; see Lemma 3.2. (c) The graph is embedded into $\mathbb{R}^3$, with all intermediate vertices located at one plane, all sources at another, and all sinks at a third one. Blue circles will become the only singular circles of the constructed Morse–Bott function. (d) The vertices are replaced by small spheres with holes, according to their degree, and the edges by tubes. This gives an immersion of a surface, with the Reeb graph of the associated height function being homeomorphic to the original graph G; see Theorem 3.4.

with all B_i. Make T_j an arborescence: choose a leaf of T_j as a root and direct the edges of T_j away from the root. Note that all non-leaf vertices of T_j are internal.

The obtained orientation has the desired properties, since any leaf of T_j either belongs to a B_i or is a leaf of G. □

Lemma 3.2. *Let G be a finite graph (allowing multiple edges and loops) without isolated vertices.*

Then there exists a tripartite (or bipartite, if G is a cycle or a path) graph $G' = (V', E')$ homeomorphic to G, with the corresponding partition

$$V' = S \cup T \cup I$$

of the set of vertices into three independent sets

$$\begin{aligned} S \cup T &= \{\, v \in V' \mid \deg v = 1 \text{ or } 2 \,\}, \\ I &= \{\, v \in V' \mid \deg v \geq 3 \,\}, \end{aligned} \tag{3.1}$$

admitting an orientation with $S \neq \emptyset$ being all sources, $T \neq \emptyset$ all sinks, and I all intermediate vertices.

Note that since G' is tripartite (or bipartite), it has no loop edges. The set I is empty when G is a cycle or a path graph.

Proof. If G is a path graph, consider $G' = K_2$: $\bullet\!\!\rightarrow\!\!\bullet$. If G is a cycle graph, consider a digraph G' with two vertices and two edges between them, oriented in the same direction: $\bullet\!\!\rightrightarrows\!\!\bullet$. In both cases, G' is bipartite.

Otherwise, smooth the graph; the resulting graph has no vertices of degree 2, though it may have loops. Consider an orientation with all non-leaf vertices being internal (Proposition 3.1). Subdivide each edge $\longrightarrow$ by two new vertices into three edges, with reverse orientation of the middle one: $\rightarrow\!\circ\!\leftarrow\!\circ\!\rightarrow$. Since G' also contains vertices of degree other than 2, it is tripartite; see Figure 1 (a), (b). $\square$

Lemma 3.3. *Let $f : M \to \mathbb{R}$ be a Morse–Bott function with k critical values. Then the function*

$$g : M \times N \xrightarrow{p} M \xrightarrow{f} \mathbb{R},$$

where p is the projection to the first factor, is also a Morse–Bott function with k critical values, and their Reeb graphs are isomorphic, $R_f = R_g$.

Indeed, $g^{-1}(y) = f^{-1}(y) \times N$, so $R_g = R_f$.

Now we can prove our main result:

Theorem 3.4. *A finite graph (allowing multiple edges and loops) is homeomorphic to the Reeb graph of a Morse–Bott function on a smooth closed n-manifold, for any $n \geq 2$.*

Proof. For simplicity, first assume $n = 2$.

Isolated vertices correspond to constant functions on connected components of the surface; we can assume now that the graph G has no isolated vertices. We can also assume that G has an orientation with the properties given in Lemma 3.2; see notation for I, S, T there.

Consider an immersion of G in $\mathbb{R}^3$ with coordinates (x, y, z) such that $I \subset \{\, z = 0 \,\}$, $S \subset \{\, z = -1 \,\}$, $T \subset \{\, z = +1 \,\}$, and edges are straight lines; see Figure 1 (c). Note that the corresponding height function increases along the direction of the edges. We will build an immersion of a surface roughly resembling a thick version of this graph, with G being the Reeb graph of the corresponding height function.

Namely, represent the vertices v of the immersed graph by spheres with boundary, embedded as shown in Figure 2 (called *atoms* in [2]), with $\deg_{\text{in}} v$ boundary components at the bottom and $\deg_{\text{out}} v$ boundary components at the top, and connect them by tubes along the edges of the graph; see Figure 1 (d). The height function is monotonous along the tubes.

This gives an immersion of a closed surface M^2 in $\mathbb{R}^3$ such that the associated height function $f : M^2 \to \mathbb{R}$ is of Morse–Bott type, and its Reeb graph coincides with G up to homeomorphism: $R_f \cong G$.

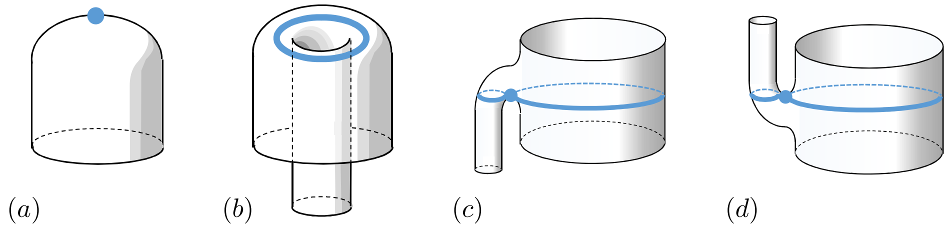

FIGURE 2. Examples of embedded surface fragments, surfaces with different number of holes at the top and bottom, used to construct the surface in Figure 1 (d). Following the pattern shown in (c) and (d), one can construct such fragments with any number of additional holes at the bottom and at the top. On all such fragments, except (b), the associated height function is of Morse type, and on (b) it is of Morse–Bott type with a singular circle. Singularities and singular levels are shown in blue.

Now, for any $n \geq 3$, consider $M^n = M^2 \times S^{n-2}$ and the composition

$$\tilde{f} \colon M^2 \times S^{n-2} \xrightarrow{p} M^2 \xrightarrow{f} \mathbb{R},$$

where p is the projection to the first factor. By Lemma 3.3, the constructed function $\tilde{f}$ is of Morse–Bott type and $R_{\tilde{f}} = R_f \cong G$. □

In the next sections, we will consider characteristics of the surface constructed in Theorem 3.4 and properties of the corresponding Morse–Bott function.

4. Further definitions and useful facts

4.1. **Morse–Bott functions with two critical values.** This case is considered in detail in a separate paper [8].

Proposition 4.1 ([8]). *Let G be a finite graph (with possible multiple edges and loops) and M a connected closed surface. Then there exists a Morse–Bott function $f \colon M \to \mathbb{R}$ with exactly two critical values and the Reeb graph R_f homeomorphic to G if and only if*

(i) *G is a path graph and M is S^2, $\mathbb{R}P^2$, or K^2 (the Klein bottle), or*
(ii) *G is a cycle graph and M is T^2 or K^2.*

The function f can be chosen as the height function associated with an immersion of M in $\mathbb{R}^3$ if and only if G is a cycle graph or M is S^2.

4.2. **Co-rank of the fundamental group.** The *co-rank* of a finitely generated group G is the maximum rank of a free homomorphic image of G. For a path-connected topological space X, consider the fundamental group $\pi_1(X)$. If it is finitely generated, as in the case of compact manifolds, then $\operatorname{corank} \pi_1(X)$ is finite. Obviously, $\operatorname{corank} \pi_1(X) \leq b_1(X)$, the first Betti number. For a surface M of genus g, it holds

$$\operatorname{corank}(\pi_1(M)) = \begin{cases} g & \text{if } M \text{ is orientable [11],} \\ \lfloor \frac{g}{2} \rfloor & \text{otherwise [5, Eq. (4.1)].} \end{cases} \tag{4.1}$$

For connected smooth closed manifolds M and N, it holds [5, Theorem 3.1]:

$$\operatorname{corank}(\pi_1(M \times N)) = \max\{ \operatorname{corank}(\pi_1(M)), \operatorname{corank}(\pi_1(N)) \}. \tag{4.2}$$

Theorem 4.2 ([7, Theorem 3.1, Proposition 3.9])**.** *Let X be a connected locally path-connected topological space and $f : X \to \mathbb{R}$ a continuous function whose Reeb graph R_f is a finite topological graph. Then for the cycle rank $b_1(R_f)$, it holds*

$$b_1(R_f) \leq \operatorname{corank}(\pi_1(X)).$$

For connected smooth closed manifolds, this inequality is tight.

5. Characteristics of the manifold and the function

In this section, we will examine the proof of Theorem 3.4 and its underlying Lemma 3.2 in more detail and, specifically, study whether the surface therein can be chosen orientable or non-orientable and having a given genus, whether the function can be chosen with a given number of critical values, and whether it can be chosen as the height function associated with an immersion of the surface in $\mathbb{R}^3$.

Accordingly, we give here extended versions of Lemma 3.2 and later Theorem 3.4.

Lemma 5.1. *Let $n, m \in \mathbb{N}$. In Lemma 3.2, the digraph $G' = (V', E')$ homeomorphic to the given graph G can be chosen such that*

(i) *$|V'| \geq n$,*

(ii) *G' has at least m edges with endpoints being a source and a sink.*

Denote $B = \{\, v \in V' \mid \deg v_i = 2 \,\}$. In addition to having the properties (i) and (ii),

(iii) *G' can be chosen such that the set B can be partitioned into pairs of adjacent vertices $\{\, s_i, t_i \,\}$, where s_i is a source and t_i a sink;*

(iv) *unless G is a tree or a cycle graph, G' can be chosen such that the set $B \setminus v_0$ can be partitioned into pairs as in (iii), for some vertex $v_0 \in B$ belonging to an undirected cycle in G'.*

Obviously, (iii) and (iv) describe two different options of the choice of G'.

Proof. (i), (ii) After constructing the graph G' as in Lemma 3.2, we can subdivide an edge by pairs of adjacent vertices of degree 2, obtaining a chain of enough edges with alternating orientation: $\to\circ\leftarrow\circ\to\circ\leftarrow\circ\to$, etc.

(iii) The construction in the proof of Lemma 3.2, with the above addition, already has this property, since we insert the vertices of degree 2 in pairs.

(iv) Let now G, and therefore G' constructed as above, has a cycle c, but is not a cycle graph. Then it has a vertex v with $\deg v \geq 3$ belonging to the cycle c; in particular, $\deg_{\text{in}} v \geq 1$ and $\deg_{\text{out}} v \geq 1$. Assume, without loss of generality, $\deg_{\text{in}} v \geq 2$:

$$\begin{array}{c} \cdots \longrightarrow v \to \cdots \\ \nearrow \end{array}$$

Subdivide the edge from the cycle c incoming to v with a vertex v_0, reversing the orientation of the new edge incident to v:

$$\begin{array}{c} \cdots \to v_0 \leftarrow v \to \cdots \\ \nearrow \end{array}$$

The new vertex v_0 with $\deg v_0 = 2$ belongs to the cycle c, and the obtained graph still satisfies the conditions (3.1); in particular, the vertex v, $\deg v \geq 3$, is still internal and v_0, $\deg v_0 = 2$, is a sink (or a source).

Note that the condition for G not to be a tree or a cycle graph is justified by the fact that a tree has no cycles to which v_0 would belong, and for a cycle graph satisfying (3.1), $|B|$ is even. $\square$

Lemma 5.2. *The surface constructed in the proof of Theorem 3.4 is orientable if the graph used there is obtained by a variant of Lemma 3.2 given in Lemma 5.1 (iii), and is non-orientable in the case of Lemma 5.1 (iv).*

The latter case is applicable only if G is not a tree or a cycle graph.

Proof. As Figure 3 shows, a pair of circle singularities (corresponding to a pair of adjacent vertices of degree 2 in the graph) can be transformed by a smooth sequence of immersions into an embedding. Thus, given that all vertices of the graph can be grouped in pairs (Lemma 5.1 (iii)), the surface in the proof of Theorem 3.4 can be transformed into an embedding of the surface in $\mathbb{R}^3$, constructed by connecting the spheres with holes (embedded at a distance from each other) by tubes (also embedded, in a non-intersecting way). Therefore, the surface is orientable. Note that the Reeb graph of the height function associated with such embedding is not homeomorphic to the given graph G, as can be seen on Figure 3 (c); this is why we did not use such a construction in the proof of Theorem 3.4.

In contrast, if after reducing all pairs of adjacent critical circles to handles one critical circle is left (Lemma 5.1 (iii)), provided that this circle is not homologically trivial, we obtain an inverted handle attached to otherwise orientable surface; see Figure 4. Thus, the whole surface is non-orientable. □

Below, we will need to add to the construction from the proof of Theorem 3.4 more surface fragments in addition to those shown in Figure 2. They will not necessarily be spheres with holes and not even be necessarily orientable.

Proposition 5.3. *Given a finite connected graph $G = (V, E)$, where $V = V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$, and a closed surface*

$$M = \Big(\bigcup_{v\in V_1} A_v\Big) \cup \Big(\bigcup_{v\in V_2} B_v\Big) \cup \Big(\bigcup_{e\in E} \tau_e\Big), \tag{5.1}$$

where all A_v and B_v are mutually disjoint surfaces with $\deg v$ boundary components, A_v being orientable and B_v non-orientable; all τ_e are mutually disjoint closed tubes; and the "vertices" A_v, B_v are interconnected by the "edges" τ_e according to the graph G:

$$\left.\begin{matrix} A_v \\ B_v \end{matrix}\right\} \cap \tau_e = \begin{cases} \text{one or two } S^1 & \text{if } e \text{ is incident to } v, \\ \emptyset & \text{otherwise} \end{cases}$$

(two S^1 above correspond to the case of a loop edge).

Then for the genus $g(M)$ of M, it holds

$$g(M) = \begin{cases} b_1(G) + \sum_{v\in V} g(A_v) & \text{if } M \text{ is orientable,} \\ 2(b_1(G) + \sum_{v\in V_1} g(A_v)) + \sum_{v\in V_2} g(B_v) & \text{otherwise.} \end{cases} \tag{5.2}$$

In the orientable case, obviously, $V = V_1$.

Proof. Since the pairwise intersections of the parts of the decomposition (5.1) are at most one-dimensional and the surface is locally compact, its Euler characteristic is additive:

$$\chi(M) = \sum_{v\in V_1} \chi(A_v) + \sum_{v\in V_2} \chi(B_v) + \sum_{e} \chi(\tau_e).$$

For the orientable surfaces A_v with $\deg v$ boundary components, $\chi(A_v) = 2 - 2g(A_v) - \deg v$, and for non-orientable B_v, it becomes $\chi(B_v) = 2 - g(B_v) - \deg v$. Since $\chi(\tau_e) = 0$, we obtain

$$\chi(M) = \sum_{v} (2 - \deg v) - 2\sum_{v\in V_1} g(A_v) - \sum_{v\in V_2} g(B_v).$$

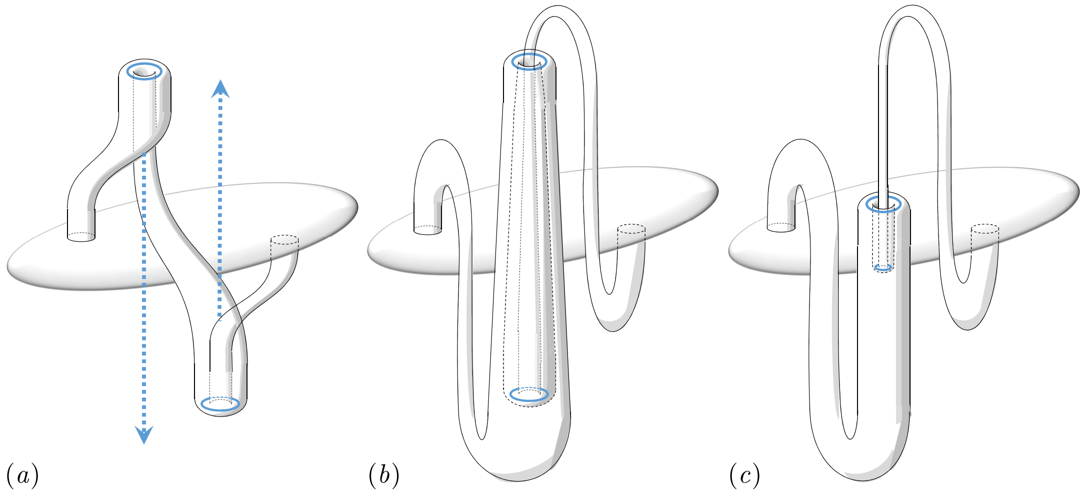

Figure 3. Transformation (a smooth family of immersions) of an immersion of the surface near a pair of adjacent critical circles into an embedding. (a) Tubes with two adjacent critical circles, connected to the rest of the surface (symbolically shown as a sphere) as explained in the proof of Theorem 3.4. Stretching the tubes along the arrows, we obtain (b), which is an embedding. Pulling one tube from the other, we obtain (c); pulling them further gives a usual handle of the surface (not shown). Note that the ends of the handle are connected from the same (outer) side of the surface.

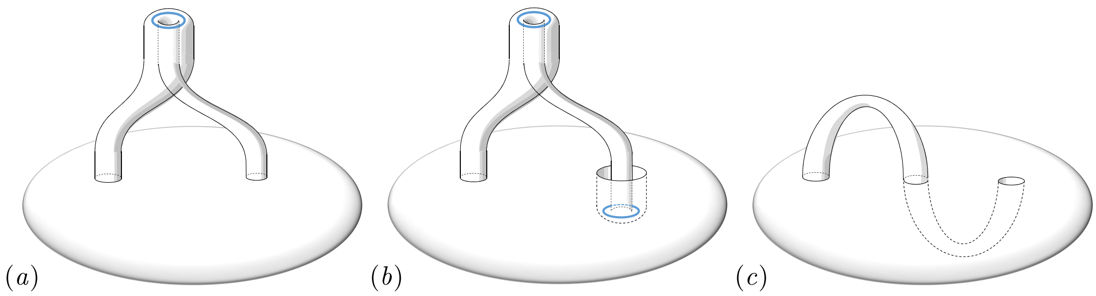

Figure 4. Similarly to Figure 3, transformation of a neighborhood of a single critical circle into an inverted handle. (a) Tubes with one critical circle similar to the standard immersion of the Klein bottle. (b) By pushing one of the tubes into the surface, we obtain a pair of two critical circles. (c) As we know from Figure 3, this is equivalent to a handle. However, this time the ends of the handle are connected from the opposite (outer and inner) sides of the surface, thus the whole surface is not orientable.

On the other hand, the cycle rank of the graph $b_1(G) = |E| - |V| + 1$, and by the handshaking lemma, $2|E| = \sum_v \deg v$, so $2b_1(G) = \sum_v (\deg v - 2) + 2$. This gives

$$\chi(M) = 2 - 2b_1(G) - 2\sum_{v \in V_1} g(A_v) - \sum_{v \in V_2} g(B_v).$$

For an orientable surface, $V = V_1$ and $\chi(M) = 2 - 2g(M)$, while for a non-orientable surface $\chi(M) = 2 - g$, which gives (5.2). □

The following theorem should be understood as two different statements, one for the orientable and one for the non-orientable case.

Theorem 5.4. *Let G be a finite connected graph (allowing multiple edges and loop edges), and $g, k \in \mathbb{Z}$, $k \geq 3$.*

Then there exists a smooth closed orientable (non-orientable) surface M of genus g and a Morse–Bott function $f : M \to \mathbb{R}$ with k critical values such that its Reeb graph R_f is homeomorphic to G if and only if G is not trivial and

$$g \geq \begin{cases} b_1(G) & \textit{(orientable case)}, \\ \max\{\, 2b_1(G), 1 \,\} & \textit{(non-orientable case)}. \end{cases} \tag{5.3}$$

The function can be chosen as the height function associated with an immersion of M in $\mathbb{R}^3$.

Proof. In one direction, suppose that the surface M (connected since G is connected) and function f in question exist. Then the Reeb graph of f is non-trivial since $k \geq 3$ implies that f is not constant. The inequality (5.3) is given by Theorem 4.2 with (4.1). Thus we only need to prove the theorem in the other direction, i.e., to prove that, provided that the graph G is not trivial, (5.3) implies the existence of a desired surface and immersion.

In the proof of Theorem 3.4, we built a closed surface M by first embedding G in $\mathbb{R}^3$ with the coordinates (x, y, z) in a special way, then replacing its vertices with fragments of the surface such as those shown in Figure 2, and replacing its edges with tubes containing no isolated critical points of the associated height function. In particular, the desired function f was the height function associated with an immersion of the constructed surface.

Without loss of generality, we can assume $G = (V, E)$ to have the properties given by Lemmas 3.2 and 5.1.

Note that if using the procedure from the proof of Theorem 3.4, we construct the desired function and embedding for some number of critical values k_0, then the theorem holds for all values of $k > k_0$. Indeed, by Lemma 5.1 (i), we can assume G to have enough vertices (most of them probably of degree 2: for example, a path graph could be represented not only as | but as something like \/\/\/\/\ as well). By a slight perturbation of their positions in the embedding of G, we can obtain an embedding of M with k critical values of the associated height function, without affecting other relevant properties of the surface and the embedding.

Similarly, provided that $k_0 \geq 3$, if the obtained surface has a genus g_0, then the theorem holds for all values of $g \geq g_0$. Indeed, by Lemma 5.1 (ii), we can assume G to have enough edges e_i connecting a source with a sink that were not affected by the above perturbation of the embedding of G, i.e., each e_i connects a vertex from $\{\, z = +1 \,\}$ with a vertex from $\{\, z = -1 \,\}$. Moreover, we can assume that the above perturbation moved vertices from $\{\, z = +1 \,\}$ only down and those from $\{\, z = -1 \,\}$ only up; thus, since $k_0 \geq 3$, there is a vertex from $\{\, z = c \,\}$ with $c \in (-1, 1)$. Subdivide some of e_i by a vertex v_i where they cross $\{\, z = c \,\}$. Construct the surface M as before, using for the new vertices v_i of degree 2 the fragments shown in Figure 5 (orientable case) or Figure 6 (non-orientable case); see Figure 7 for more details. By Proposition 5.3, this increases the genus of M, without affecting other relevant properties.

Therefore, we only need to construct, using the procedure from the proof of Theorem 3.4, the required function, via a suitable immersion, for some $k_0 \leq 3$ and g given by the right-hand side of (5.3).

For the orientable case, the original construction given in Theorem 3.4 already has all the desired properties. Indeed, the construction given in Lemma 3.2 corresponds to the option (iii)

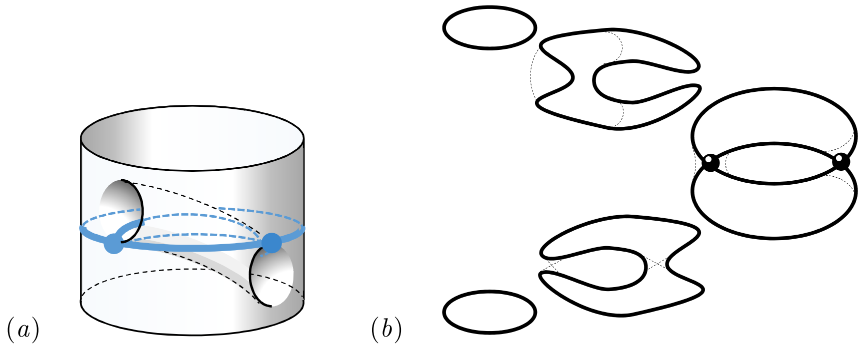

FIGURE 5. An embedding or a surface fragment used to construct the surface M in addition to those shown in Figure 2 (orientable case). In contrast to Figure 2, this fragment has nonzero genus, while the Reeb graph of the associated height function f has zero cycle rank; therefore, this fragment increases the genus of M by 1, without increasing $b_1(R_f)$. (b) Evolution of the level sets of the height function f. The thin dashed lines show how the next (from bottom to top) level is obtained from the current one: for example, self-gluing the second level in two places gives the third level. These level sets are shown in a different way in Figure 7 (a).

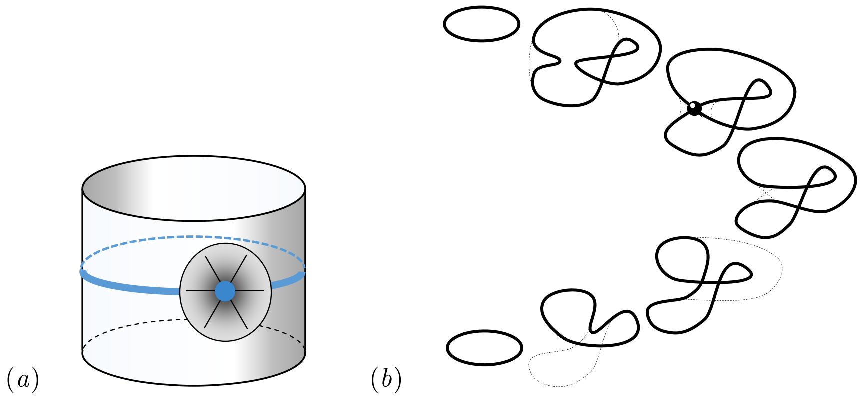

FIGURE 6. Same as Figure 5, non-orientable case: the Boy surface—an immersion of $\mathbb{R}P^2$ in $\mathbb{R}^3$. (a) Symbolic visual representation (only symbolizing that "something happened there"); in more detail the immersion is shown in [2, Fig. 2.21 (a)]. (b) Evolution of the level sets. These level sets are shown in a different way in Figure 7 (c). Figure adapted from [2, Example 5, page 62].

of Lemma 5.1; then Lemma 5.2 gives that M is orientable. Thus, since all surface fragments representing the vertices of the graph were spheres with boundary (of genus zero), Proposition 5.3 gives the genus $g(M) = b_1(G)$. By construction, the associated height function has at most three critical values.

In the rest of the proof, we consider the non-orientable case. Unless G is a tree or a cycle graph, the construction from Theorem 3.4 with Lemma 5.1 (iv) has the desired properties, given, again, by Lemma 5.2 and Proposition 5.3.

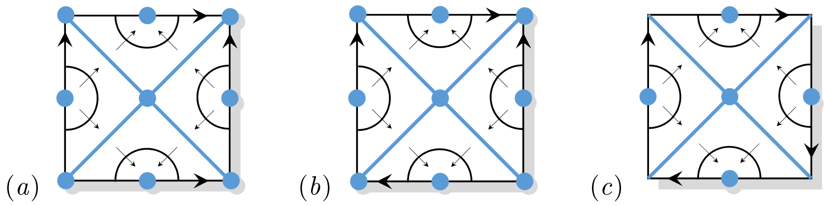

FIGURE 7. Morse functions on three surfaces with all critical points of index 1 lying on the same contour (blue). The surfaces are shown as a square with the sides identified according to the arrows; thin arrows indicate the gradient direction. (a) Torus T^2 shown in Figure 5. (b) For completeness, the Klein bottle K^2 (note the opposite orientation of the bottom side of the square). We do not use this construction in this paper, since it would add 2 to the genus if used in a similar way as (a) and (c). (c) The projective plane $\mathbb{R}P^2$ shown in Figure 6.

For G being a tree or a cycle graph, the right-hand side of (5.3) gives $g = 1$. For a tree, we first construct the embedding of the graph G as in the orientable case. As above, we can assume that there is an edge between $\{ z = -1 \}$ and $\{ z = +1 \}$ (substituting some edge $|$ with $\backslash\!/\!\backslash$ if needed). Subdivide this edge with a vertex of degree 2, for which, when constructing the surface, use the fragment from Figure 6. The resulting non-orientable surface has genus one by Proposition 5.3. If the graph already had vertices at $\{ z \neq \pm 1 \}$, then the new vertex is placed at the same z. By construction, we obtain three critical levels.

Finally, for a cycle graph, we construct a torus as in the orientable case but using the immersion as in Figure 8 (a), and twist one of the tubes as in Figure 8 (b), obtaining an immersion of the Klein bottle K^2 with the desired properties. □

For the sake of completeness, Theorem 5.4 and Proposition 4.1 can be combined in a criterion for a given graph G to be realizable on a given surface M as the Reeb graph of a function with a given number k of critical values (again, as a combination of two statements, for the orientable and for the non-orientable case):

Theorem 5.5. *Let G be a finite connected graph (allowing multiple edges and loop edges), and $g, k \in \mathbb{Z}$.*

Then there exists a smooth closed orientable (non-orientable) surface M of genus g and a Morse–Bott function $f : M \to \mathbb{R}$ with k critical values such that its Reeb graph R_f is homeomorphic to G if and only if

$$g \geq \begin{cases} b_1(G) & \textit{(orientable case)} \\ \max\{ 2b_1(G), 1 \} & \textit{(non-orientable case)} \end{cases}$$

and $k = 1$ if G is trivial, otherwise

$$k \geq \begin{cases} 2 & \textit{if the conditions } (5.5) \textit{ below hold,} \\ 3 & \textit{otherwise,} \end{cases} \tag{5.4}$$

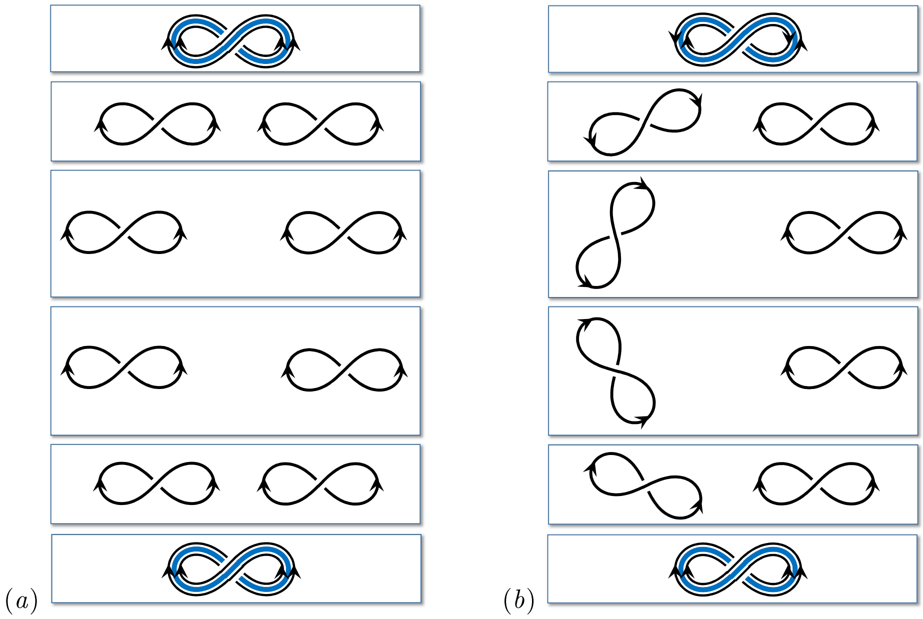

FIGURE 8. (a) An immersion i of a torus T^2 in $\mathbb{R}^3$ with coordinates (x, y, z). The rectangles represent horizontal planes $\{ z = \text{const} \}$ at different height, with the intersections (8-shaped) with $i(T^2)$ shown. In blue color, the images of the singular circles (minimum and maximum) of the associated Morse–Bott height function $f(x, y, z) = z$ are shown, along with a nearby level set each. Arrows show the evolution of an orientation on the contours; the orientation is consistent. (b) A similar immersion of the Klein bottle K^2. The twist (left) results in inconsistent orientation, as the evolution of the arrows, seen clock-wises starting from the top, shows; therefore, the obtained circle bundle over S^1 is non-orientable, i.e., K^2. Figure and parts of its description are borrowed from [8].

the conditions for $k = 2$ *being:*

$$g = \begin{cases} \left.\begin{array}{ll} 0, \ M = S^2 & \textit{(orientable case)} \\ 1, \ M = \mathbb{R}P^2 & \textit{(non-orientable case)} \\ 2, \ M = K^2 & \textit{(non-orientable case)} \end{array}\right\} & \textit{and } G \textit{ is a path graph, or} \\ \left.\begin{array}{ll} 1, \ M = T^2 & \textit{(orientable case)} \\ 2, \ M = K^2 & \textit{(non-orientable case)} \end{array}\right\} & \textit{and } G \textit{ is a cycle graph.} \end{cases} \tag{5.5}$$

The function f can be chosen as the height function associated with an immersion of M in $\mathbb{R}^3$ if and only if G is not trivial and, in the non-orientable case, either $k \neq 2$ or G is a cycle graph (in particular, $M = K^2$, i.e., $g = 2$).

Theorem 5.5 can be formulated in terms of a given surface M instead of a given genus g; this slightly simplifies the expression (5.3) by removing the need for the $\max\{\cdot\}$. Next, (4.1) allows further simplifying the expression (5.3) by considering the co-rank of the fundamental group instead of genus; this may potentially allow for generalizations to higher dimensions. In the following form of the theorem, for simplicity we omit the part about the number k of critical values, which can be borrowed exactly from Theorem 5.5, substituting in (5.4) a reference to (5.3) with a reference to (5.6):

Theorem 5.6. *Let G be a finite graph (with possible multiple edges and loop edges) and M a smooth closed surface.*

Then there exists a Morse–Bott function $f : M \to \mathbb{R}$ such that its Reeb graph R_f is homeomorphic to G if and only if

$$\operatorname{corank}(\pi_1(M)) \geq b_1(G). \tag{5.6}$$

The function f can be chosen as the height function associated with an immersion of M in $\mathbb{R}^3$ if and only if G has no isolated vertices.

Theorem 5.5 can be easily generalized to higher dimensionality. Omitting less important details such as the number of critical values and the possibility for the function to be the height function associated with an immersion, we can generalize this theorem as follows:

Theorem 5.7. *Let G be a finite connected graph (allowing multiple edges and loop edges) and $n, c \in \mathbb{Z}$, $n \geq 2$.*

Then there exists a smooth closed orientable (non-orientable) n-manifold M with

$$\operatorname{corank}(\pi_1(M)) = c$$

and a Morse–Bott function $f : M \to \mathbb{R}$ such that its Reeb graph R_f is homeomorphic to G if and only if

$$c \geq b_1(G).$$

Proof. In one direction, given that such a manifold M and function f exist, Theorem 4.2 gives $c \geq b_1(G)$.

In the other direction, given $c \geq b_1(G)$ and $k \geq 2$, we need to construct a manifold M^n and a Morse–Bott function $f : M \to \mathbb{R}$ on it with $\operatorname{corank}(\pi_1(M^n)) = c$ and $R_f \cong G$.

For $n = 2$, the proof is given in Theorem 5.5 with $g = c$ (orientable case) or $g = 2c + 1$ (non-orientable case).

For $n = 3$ and $c = 0$, consider $M = S^3$, a sphere. Since $b_1(G) = 0$, the graph is a tree. By [14, Theorem 6.4], on S^3 there exists a Morse function $f : S^3 \to \mathbb{R}$ such that $R_f \cong G$.

Let now $n \geq 3$ with $c \geq 1$ if $n = 3$. Consider $M = M^2 \times S^{n-2}$, with the surface M^2 being orientable of genus $g = c$ in the orientable case, and non-orientable of genus $g = 2c+1$ in the non-orientable case; in both cases, $\operatorname{corank}(\pi_1(M^2)) = c$. Since either $\operatorname{corank}(\pi_1(S^{n-2})) = 0$ (for $n \geq 4$) or $\operatorname{corank}(\pi_1(M^2)) \geq 1$ (for $n = 3$), Equation (4.2) gives $c = \operatorname{corank}(\pi_1(M)) = \operatorname{corank}(\pi_1(M^2)$.

Since $g \geq b_1(G)$, by Theorem 5.5, there is a Morse–Bott function $\tilde{f} : M^2 \to \mathbb{R}$ such that $R_{\tilde{f}}$ is homeomorphic to G. By Lemma 3.3, the composition

$$f : M^2 \times S^1 \xrightarrow{p} M^2 \xrightarrow{\tilde{f}} \mathbb{R},$$

where p is the projection to the first factor, is a Morse–Bott function, with $R_f = R_{\tilde{f}} \cong G$.

In the non-orientable case, choose the surface M^2 to be non-orientable of genus $g = 2c+1$; then $\operatorname{corank}(\pi_1(M^2) = c$. As above, since $g \geq 2b_1(G)+1$, by Theorem 5.5 and Lemma 3.3, we construct a Morse–Bott function $f : M^2 \times S^1 \to \mathbb{R}$ with k critical values and $R_f = R_{\tilde{f}} \cong G$. □

Another way of studying the higher-dimensional case is to consider the same construction we used in the proof of Theorem 3.4, but with n-dimensional spheres with holes ($S^n \setminus \bigcup B_i^n$) instead of 2-dimensional ones shown in Figure 2, represening the vertices of the graph, similarly connected by n-dimensional "handles" ($S^{n-1} \times \mathbb{R}$) representing the edges of the graph. Indeed, in Figure 2, we construct an embedding of a sphere with holes as an evolution of a number of S^{n-1} (in our case, $n = 2$) along the height dimension (say, moving them from bottom up), merging and splitting them all at the same (critical) height, see Figure 2 $(a), (c), (d)$, or reversing the direction of the "movement" with suitable resizing, see Figure 2 (b). The same can be done for $n \geq 2$, similarly obtaining roughly a thick version of the graph. We leave the study of this construction to the future work.

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