Compact and locally dense leaves of a closed one-form foliation

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Abstract

We study a foliation defined by a possibly singular smooth closed one-form on a connected smooth closed orientable manifold. We prove two bounds on the total number of homologically independent compact leaves and of connected components of the union of all locally dense leaves, which we call minimal components. In particular, we generalize the notion of minimal components, previously used in the context of Morse form foliations, to general foliations. Finally, we give a condition for the form foliation to have only closed leaves (closed in the complement of the singular set).

Key words: closed one-form, foliation, compact leaves, locally dense leaves, minimal components
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1. Introduction

Let ω be a smooth closed one-form on a connected smooth closed orientable manifold M and $\operatorname{Sing}\omega$ the set of its singularities. This form defines on $M \setminus \operatorname{Sing}\omega$ a codimension-one foliation $\mathcal{F}_\omega$. This type of foliations is important in applications to physics, e.g., in supergravity theory [1, 2].

Smooth closed one-forms define an important class of foliations: foliations without holonomy; moreover, any codimension-one foliation without holonomy is topologically equivalent to a foliation defined by a smooth closed one-form [3]. A subset of smooth closed one-forms, Morse forms (locally the differential of a Morse function), is well-studied.

A leaf of a codimension-one foliation is either *proper* (locally closed, hence it is a regular submanifold), *locally dense* (its closure has non-empty interior), or *exceptional* (its closure is known to be transversally homeomorphic to a Cantor set). In particular, a compact leaf is proper.

The number of different leaves of each kind, usually up to some equivalence relation, or objects related with such equivalence classes, is an important topological invariant of a foliation. In this paper, we will study the number of locally

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dense and compact leaves of a closed one-form foliation up to some topological and homological equivalence, respectively.

One construction used (not in this paper) to count equivalence classes of leaves of a foliation (defined on the whole arbitrary manifold M) is *minimal sets*: minimal non-empty closed saturated (i.e., consisting of whole leaves) subsets of M. The number of exceptional minimal sets of a codimension-one foliation on a compact manifold is finite; moreover, for arbitrary $k \in \mathbb{N}$ there exists a foliation on a compact manifold with k exceptional minimal sets [4, Theorem 4.1.3]. This number is finite even for a C^0-foliation on a closed manifold [5, Lemma 2.13].

However, each compact leaf is a minimal set in itself, the union of all compact leaves of a closed one-form foliation being open, so if a foliation has a compact leaf, then the number of its minimal sets is infinite. It is even worse for locally dense leaves: a minimal set contains a locally dense leaf L only if $\overline{L} = M$. So minimal sets are not a suitable tool for studying compact and locally dense leaves.

Another construction is connected components of the union of leaves of each type. For a Morse form foliation $\mathcal{F}_\omega$, connected components of the union of locally dense leaves are called *minimal components* [6] and connected components of the union of compact leaves are called *maximal components* [7], or in the case of 2-surfaces, *periodic components* [8].

A Morse form foliation contains only closed (in $M \setminus \operatorname{Sing}\omega$) and locally dense leaves [6, 9]. The number $m(\omega)$ of its minimal components, the number $M(\omega)$ of its maximal components, and the number of its singularities are finite [7, 10]; moreover (Corollary 2.2):

$$
\begin{aligned}
2m(\omega) + M(\omega) &\le b_1(M) + |\operatorname{Sing}\omega| - 1,\\
m(\omega) + M(\omega) &\le b_1'(M) + |\operatorname{Sing}\omega| - 1,
\end{aligned}
$$

where $b_1(M)$ is the Betti number and $b_1'(M)$ is the co-rank of the fundamental group $\pi_1(M)$; the second inequality is exact for each M. The number of closed (in $M \setminus \operatorname{Sing}\omega$) but non-compact leaves of a Morse form foliation is also finite. However, whereas we will show below that $m(\omega)$ can be generalized to arbitrary smooth closed one-form, these inequalities cannot be generalized because both $\operatorname{Sing}\omega$ and $M(\omega)$ can generally be infinite, as in Example 3.1 below.

For the study of compact leaves, homology theory provides more useful tools: obviously, for a form on a compact manifold, the number $c(\omega)$ of homologically independent compact leaves of $\mathcal{F}_\omega$ is finite: $c(\omega) \le b_1(M)$, the first Betti number. For a Morse form foliation, the number $c(\omega)$ of homologically independent compact leaves and the number $m(\omega)$ of minimal components are related:

$$
\begin{aligned}
2m(\omega) + c(\omega) &\le b_1(M), && \text{[7, Theorem 3.1]}\\
m(\omega) + c(\omega) &\le b_1'(M). && \text{[10, Theorem 3]}
\end{aligned}
$$

In this paper, we generalize the notion of a minimal component to arbitrary foliations and show that these two bounds hold for arbitrary smooth closed one-forms.

The paper is organized as follows. In Section 2, we give necessary definitions and known facts concerning smooth closed one-form foliations, Morse form foliations, the co-rank $b_1'(M)$ of the fundamental group $\pi_1(M)$, close cohomologous one-forms, and $\mathcal{F}$-saturated sets. In Section 3, we generalize the notion of a minimal component from Morse form foliations to arbitrary smooth closed one-form foliations and study its properties. In Section 4, we introduce extended minimal components, which are open sets in one-to-one correspondence with minimal components. In Section 5, we prove our main theorem: the bounds on $c(\omega)$ and $m(\omega)$ in terms of $b_1'(M)$ and the first Betti number $b_1(M)$. Finally, in Section 6, we apply this theorem to obtain a condition for compactifiability of a smooth closed one-form foliation in terms of $b_1'(M)$.

2. Definitions and useful facts

Unless stated otherwise, we will consider a connected smooth closed orientable n-dimensional manifold M.

2.1. Smooth closed one-form foliation

Consider a smooth closed one-form ω on M; its singular set $\operatorname{Sing}\omega = \{\, x \in M \mid \omega_x = 0 \,\}$ is closed in M. This form defines a codimension-one foliation $\mathcal{F}_\omega$ on $M \setminus \operatorname{Sing}\omega$.

A leaf of a codimension-one foliation is either *proper* (or *embedded*: locally closed; in particular, leaves closed in $M \setminus \operatorname{Sing}\omega$ are proper), *locally dense* (its closure has non-empty interior), or *exceptional*.

A leaf of $\mathcal{F}_\omega$ closed in $M \setminus \operatorname{Sing}\omega$ is "compactified" by $\operatorname{Sing}\omega$ in the sense that $L \cup \operatorname{Sing}\omega$ is compact. Such leaves are just closed leaves in terms of the foliation defined on $M \setminus \operatorname{Sing}\omega$. However, to avoid repetitive reminding of that closeness is meant in $M \setminus \operatorname{Sing}\omega$ and not in M, we will call such leaves *compactifiable*; all other leaves are *non-compactifiable*. A foliation is called *compactifiable* if all its leaves are compactifiable [11].

In particular, compact leaves, i.e., leaves closed even in M, are compactifiable. The union of all compact leaves is open [12, Lemma 3.1].

In this paper, we study locally dense and compact leaves of $\mathcal{F}_\omega$.

Denote by $H_\omega \subseteq H_{n-1}(M)$ the subgroup generated by the homology classes of all compact leaves, and let

$$c(\omega) = \operatorname{rk} H_\omega,$$

which is the maximum number of homologically independent leaves of $\mathcal{F}_\omega$:

Theorem 2.1 ([13, Theorem 3.1]**).** *Let $H_\omega \neq 0$. Then there exists a basis e of H_ω consisting of homology classes of leaves: $e = \{\, [L_1], \dots, [L_{c(\omega)}] \,\}$, $L_i \in \mathcal{F}_\omega$.*

The *rank* of the form is $\operatorname{rk}\omega = \operatorname{rk}_{\mathbb{Q}} \operatorname{im}[\omega]$, where

$$[\omega]\colon H_1(M) \to \mathbb{R}$$

is the integration map. Obviously, $0 \le \operatorname{rk}\omega \le b_1(M)$, the Betti number. A form ω is exact if and only if $\operatorname{rk}\omega = 0$.

For a subgroup $H \subseteq H_{n-1}(M)$, denote

$$H^\ddagger = \{\, z \in H_1(M) \mid z \cdot H = 0 \,\},$$

where $\cdot$ is the cycle intersection. Obviously, $G \subseteq H$ implies $H^\ddagger \subseteq G^\ddagger$.

Proposition 2.1 ([13, Theorem 4.2])**.** *Let* $H_\omega^\ddagger \subseteq \ker[\omega]$. *Then:*

(i) $\mathcal{F}_\omega$ *is compactifiable;*

(ii) $\operatorname{rk}\omega \le c(\omega)$*;*

(iii) *if* $c(\omega) \ge 1$, *then, for any* $k = 1, \dots, c(\omega)$, *in any neighborhood of* ω, *there exists a smooth closed one-form* ω' *with* $\operatorname{rk}\omega' = k$ *defining the same foliation,* $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$.

2.2. Morse form foliation

A *Morse form* ω is a smooth closed one-form that is locally the differential of a Morse function. This is an important special case, since the set of Morse forms is open and dense in the space of smooth closed one-forms [12, Lemma 2.1].

The structure of a Morse form foliation is well studied [7, 14]; it is much simpler than the structure of a general smooth closed one-form foliation. In particular, its singular set $\operatorname{Sing}\omega$ is finite, and it has a finite number of non-compact compactifiable leaves. A Morse form has no exceptional leaves; moreover:

Proposition 2.2 ([6, 9])**.** *Leaves of a Morse form foliation are either compactifiable or locally dense.*

This means that non-compactifiable leaves of $\mathcal{F}_\omega$ are exactly locally dense leaves; in particular, $\mathcal{F}_\omega$ has no exceptional leaves or proper non-closed (in $M \setminus \operatorname{Sing}\omega$) leaves.

For Morse forms, we have a criterion; cf. Proposition 2.1 (i):

Proposition 2.3 ([11, Theorem 7])**.** *A Morse form foliation* $\mathcal{F}_\omega$ *is compactifiable if and only if* $H_\omega^\ddagger \subseteq \ker[\omega]$.

Definition 2.1 ([6])**.** *A* minimal component $\mathcal{C}$ *of a Morse form foliation* $\mathcal{F}_\omega$ *is a connected component of the union of all non-compactifiable leaves.*

Note that minimal components are not what in the theory of foliation is called minimal sets. By Proposition 2.2, a minimal component of a Morse form is a connected component of the union of all locally dense leaves. Minimal components of Morse forms have nice properties:

Proposition 2.4 ([9, Proposition 1.8; §2], [6, p. 155])**.** *Each minimal component of a Morse form foliation is open, and each locally dense leaf is dense in its minimal component.*

Denote by $m(\omega)$ the number of minimal components of $\mathcal{F}_\omega$, and recall that $c(\omega)$ is the maximum number of homologically independent compact leaves. Denote by $b_1(M)$ the first Betti number and by $b_1'(M)$ the co-rank of the fundamental group $\pi_1(M)$.

In this paper, we generalize to smooth closed one-forms the following three facts known for Morse forms:

Theorem 2.2 ([7, Theorem 3.1])**.** *For a Morse form ω, it holds that:*

$$2m(\omega) + c(\omega) \le b_1(M). \tag{1}$$

Theorem 2.3 ([10, Theorem 3])**.** *For a Morse form ω, it holds that:*

$$0 \le m(\omega) + c(\omega) \le b_1'(M) \tag{2}$$

and all intermediate values are reached on a given M; in particular, the bounds are exact.

Corollary 2.1. *For a Morse form ω, if $c(\omega) = b_1(M)$ or $c(\omega) = b_1'(M)$, then the foliation $\mathcal{F}_\omega$ is compactifiable.*

Proof. Under these conditions, the above theorems give $m(\omega) = 0$ and by Proposition 2.2, all leaves of $\mathcal{F}_\omega$ are compactifiable. □

However, as mentioned in Section 1, the following facts do not generalize:

Corollary 2.2. *For a Morse form ω with $\operatorname{Sing}\omega \neq \emptyset$, it holds that:*

$$2m(\omega) + M(\omega) \le b_1(M) + |\operatorname{Sing}\omega| - 1, \tag{3}$$

$$m(\omega) + M(\omega) \le b_1'(M) + |\operatorname{Sing}\omega| - 1, \tag{4}$$

where $M(\omega)$ is the number of maximal components of $\mathcal{F}_\omega$ (connected components of the union of compact leaves). The second bound is exact for each M.

Proof. The *foliation graph* of a Morse form is constructed of the maximal components as edges and the connected components of the rest of M as vertices [7]. For its circuit rank $m(\Gamma) = M(\omega) - p + 1$, where p is the number of vertices, [7, Theorem 2.1] states $m(\Gamma) = c(\omega)$. It is known that each vertex contains a singularity; thus $p \le |\operatorname{Sing}\omega|$. With this, (1) and (2), respectively, give the desired bounds.

The second bound is exact for a generic form, i.e., a form for with $p = |\operatorname{Sing}\omega|$, defining a compactifiable foliation [10, Proposition 5]. Generic Morse forms are dense in each cohomology class [15, Lemma 9.2]. □

The bounds (1) and (3) are not exact: for an n-torus T^n, we have $b_1(T^n) = n$ and $b_1'(T^n) = 1$ (Example 2.1); then (2) gives $m(\omega) \le 1$ and (4) gives $M(\omega) \le |\operatorname{Sing}\omega|$, which makes equality in (1) and (3) impossible for large n.

2.3. Co-rank of the fundamental group

Relations between the co-rank $b_1'(M)$ of the fundamental group $\pi_1(M)$, i.e., the maximum rank of its free quotient group, and the first Betti number $b_1(M)$ were studied in [16, 17]. Obviously, $b_1'(M) \le b_1(M)$; more specifically:

Theorem 2.4 ([17, Theorem 4.1])**.** *Let $b', b, n \in \mathbb{Z}$. There exists a connected smooth closed n-manifold M with $b_1'(M) = b'$ and $b_1(M) = b$ if and only if*

$$\begin{aligned} n = 0\colon &\quad b' = b = 0; \\ n = 1\colon &\quad b' = b = 1; \\ n = 2\colon &\quad 0 \le b \text{ and } b' = \left\lfloor \tfrac{b+1}{2} \right\rfloor; \\ n \ge 3\colon &\quad b' = b = 0 \text{ or } 1 \le b' \le b. \end{aligned}$$

The manifold can be chosen orientable if and only if $n \ne 2$ or b is even.

Some methods of calculating $b_1'(M)$ can be found in [17, 18, 19]; in particular:

(i) $b_1'(S^1) = 1$;

(ii) $b_1'(M \# N) = b_1'(M) + b_1'(N)$, the connected sum;

(iii) $b_1'(M \times N) = \max\{ b_1'(M), b_1'(N) \}$, the direct product.

Example 2.1. *Examples of calculating $b_1'(M)$ include:*

(1) *For n-torus $T^n = S^1 \times \cdots \times S^1$,* (i) *and* (iii) *give $b_1'(T^n) = 1$.*

(2) *For a closed orientable surface $M_g^2 = \#_{i=1}^g T^2$,* (ii) *gives $b_1'(M_g^2) = g$.*

(3) *For $M = \#_{i=1}^p (S^n \times S^1)$, $n \ge 2$,* (ii) *and* (iii) *give $b_1'(M) = p$.*

2.4. Close cohomologous forms

Foliations defined by close one-forms can have very different topological structure: for example, a form with rational coefficients on a torus defines a compact foliation, whereas a close form with an irrational coefficient defines a winding, i.e., a minimal foliation.

However, foliations of smooth closed one-forms that are both close and cohomologous have, in some sense, similar topology. For example, compact leaves are stable under small perturbations of the form in its cohomology class. In particular, denote by $F(\Omega)$ the space of smooth closed one-forms representing a class $\Omega \in H^1(M, \mathbb{R})$; then:

Proposition 2.5 ([12, Theorem 3.1])**.** *Let ω be a smooth closed one-form. Then there exists a neighborhood $\mathcal{U}(\omega) \subseteq F([\omega])$ such that $H_\omega \subseteq H_{\omega'}$ for any $\omega' \in \mathcal{U}(\omega)$.*

The following statement shows that facts concerning Morse form foliations are useful in the study of arbitrary smooth closed one-form foliations:

Proposition 2.6 ([20, Ch. 2, Theorem 1.25])**.** *On a closed manifold M, the set of Morse forms is open and dense in each cohomology class $\Omega \in H^1(M, \mathbb{R})$.*

2.5. $\mathcal{F}$-saturated sets

Given a foliation $\mathcal{F}$ on M, a subset $X \subseteq M$ is *$\mathcal{F}$-saturated* (or *invariant*) if it is the union of some leaves of $\mathcal{F}$. Obviously, the complement of a saturated set, as well as the union, the intersection, and the difference of two saturated sets, are also saturated.

Proposition 2.7 ([21, Proposition 1.3])**.** *If $X \subseteq M$ is $\mathcal{F}$-saturated, then so are $\overline{X}$ and* $\operatorname{int}(X)$.

In particular, the closure of each leaf is saturated; the boundary ∂X of a saturated set is also saturated.

A *minimal set* μ of a foliation is a closed non-empty saturated subset of M that has no proper subset with these properties [22, Definition 4.1.1].

For any leaf $L \subseteq \mu$, it holds that $\mu = \overline{L}$. For a closed leaf, a minimal set exists but coincides with it. Since $\partial\overline{L}$ is closed and, by Proposition 2.7, saturated, a leaf belongs to a minimal set only if $\partial\overline{L} = \emptyset$ or $\operatorname{int}(\overline{L}) = \emptyset$, so a locally dense leaf lies in a minimal set only if $\mu = \overline{L} = M$, i.e., the foliation is minimal. Note that in the case of a form foliation $\mathcal{F}_\omega$, here M is understood as the complement of $\operatorname{Sing}\omega$, and closeness is understood in this complement.

Thus the notion of a minimal set is mostly useful for the study of exceptional leaves. For the study of compact leaves, maximal components [7] and homology classes are used instead, whereas for locally dense leaves, minimal components discussed in Section 3 below play a similar role.

3. Minimal components of foliations

We will generalize the notion of a *minimal component* from the theory of Morse form foliations to general foliations in a way that preserves its usefulness for studying locally dense leaves.

This notion is not to be confused with that of a *minimal set*, which typically cannot contain locally dense leaves as discussed in Section 2.5.

3.1. Definition of a minimal component of a foliation

In the case of Morse forms, one can consider several equivalent statements as a definition of a minimal component:

Proposition 3.1. *For a Morse form foliation, the following definitions of a minimal component $\mathcal{C}$ are equivalent:*

(i) *$\mathcal{C}$ is a connected component of the union of all non-compactifiable leaves (Definition 2.1).*

(ii) *$\mathcal{C}$ is a connected component of the union of all locally dense leaves.*

(iii) *$\mathcal{C}$ is a minimal non-empty open saturated set (which justifies the term* minimal component*).*

Proof. (i) $\Leftrightarrow$ (ii): By Proposition 2.2, non-compactifiable leaves of a Morse form foliation are exactly its dense leaves.

(ii) $\Rightarrow$ (iii): By Proposition 2.4, $\mathcal{C}$ is open. Suppose $\mathcal{C}$ is not minimal, i.e., there is an open saturated set $\mathcal{C}' \subsetneq \mathcal{C}$. Then $\partial\mathcal{C}' \cap \mathcal{C} \neq \emptyset$. By Proposition 2.7, $\partial\mathcal{C}'$ is $\mathcal{F}_\omega$-saturated, so there is a locally dense leaf $L \subseteq \partial\mathcal{C}' \cap \mathcal{C}$ with $\operatorname{int}(\overline{L}) \subset \overline{L} \subset \partial\mathcal{C}'$, whereas the boundary of an open set has empty interior; a contradiction.

(iii) $\Rightarrow$ (ii): If there exists a leaf $L \subset \mathcal{C}$ that is not locally dense, then $\mathcal{C} \setminus \overline{L}$ is non-empty, open, and saturated, i.e. $\mathcal{C}$ is not minimal. □

However, in the case of general foliations, these properties are not equivalent and are not equally suitable to serve as a definition of a minimal component—a tool for studying locally dense leaves:

Variant (i) from Proposition 3.1: This is the original definition from [6]. However, apart from locally dense leaves, $\mathcal{C}$ would contain proper non-closed and exceptional leaves, which makes it not suitable for the study of locally dense leaves.

Variant (ii) from Proposition 3.1: $\mathcal{C}$ is a saturated set containing only locally dense leaves. In a sense it is minimal, since it is a connected component of the union of all such leaves. However, as Example 3.1 below shows, it is not guaranteed to be open.

Variant (iii) from Proposition 3.1: While it is nice that $\mathcal{C}$ would be guaranteed to be open and the term *minimal component* would be well-justified, Example 3.1 shows that such a set does not always exist.

As a generalization of the notion of minimal component, we prefer the guarantee of existence (ii) over the guarantee of openness (iii):

Definition 3.1. *A* minimal component $\mathcal{C}$ *is a connected component of the union of all locally dense leaves.*

Obviously, minimal components are $\mathcal{F}$-saturated sets.

Lemma 3.1. *If a leaf $L \subseteq \mathcal{C}$ (a minimal component), then:*

(i) $\mathcal{C} \subseteq \operatorname{int}(\overline{L})$*; and*

(ii) $\overline{\mathcal{C}} = \overline{L}$.

In particular, each locally dense leaf L is dense in its minimal component: $\mathcal{C} \subseteq \overline{L}$, and for each locally dense leaf L it holds that $L \subset \operatorname{int}(\overline{L})$.

Proof. (i) Suppose $\mathcal{C} \not\subseteq \operatorname{int}(\overline{L})$. Since $\mathcal{C}$ is connected and $\mathcal{C} \cap \overline{L} = \emptyset$, we obtain $\mathcal{C} \cap \partial\overline{L} \neq \emptyset$. By Proposition 2.7, the boundary $\partial\overline{L}$ is $\mathcal{F}$-saturated. Since both $\mathcal{C}$ and $\partial\overline{L}$ are saturated, there exists a locally dense leaf $L' \subset \mathcal{C}$ such that $L' \subseteq \partial\overline{L}$, so $\emptyset \neq \operatorname{int}(\overline{L'}) \subset \overline{L'} \subseteq \partial\overline{L}$, whereas the boundary of a closed set has empty interior; a contradiction.

(ii) $L \subseteq \mathcal{C}$ gives $\overline{L} \subseteq \overline{\mathcal{C}}$, whereas (i) implies $\overline{\mathcal{C}} \subseteq \overline{L}$. Thus $\overline{\mathcal{C}} = \overline{L}$. □

Proposition 3.2. *Locally dense leaves L_1, L_2 belong to the same minimal component if and only if $\overline{L}_1 = \overline{L}_2$.*

PROOF. For $L_1, L_2 \subset \mathcal{C}$, Lemma 3.1 (ii) implies $\overline{L}_1 = \overline{L}_2 = \overline{\mathcal{C}}$. Now, let L_1, L_2 be locally dense leaves such that $\overline{L}_1 = \overline{L}_2$, and denote $\mathcal{C}_i \supseteq L_i$ the corresponding minimal components. By the same lemma, we have $\overline{\mathcal{C}}_1 = \overline{\mathcal{C}}_2$, thus $\mathcal{C}_1 \cup \mathcal{C}_2$ is connected, which for connected components implies $\mathcal{C}_1 = \mathcal{C}_2$. □

Thus a minimal component $\mathcal{C}$ is the union of locally dense leaves that share a common closure. Namely, consider an equivalence relation on $\mathcal{F}$ such that $L_1 \sim L_2$ if and only if $\overline{L}_1 = \overline{L}_2$; this relation preserves the property of a leaf to be locally dense. Then a minimal component $\mathcal{C} \supset L$ can be defined as an area covered by an equivalence class of locally dense leaves:

$$\mathcal{C} = \bigcup_{\overline{L'} = \overline{L}} L'.$$

So a minimal component consists of, in a sense, similar leaves, being a minimal subset containing these similar leaves.

3.2. Non-open minimal components

Lemma 3.2. *A minimal component $\mathcal{C}$ of a foliation is either open or has empty interior.*

PROOF. If C is not open, then $\mathcal{C} \cap \partial\mathcal{C} \neq \emptyset$. Thus there exists a leaf $L \subseteq \mathcal{C} \cap \partial\mathcal{C}$, since both sets are saturated. Since L is dense in $\mathcal{C}$, we obtain $\operatorname{int}(\mathcal{C}) = \emptyset$. □

Along the lines of [23, Example 2.10], it is easy to construct a non-closed manifold with a smooth closed one-form foliation that has a non-open minimal component. In the rest of this section we will show that, in contrast to the case of Morse forms foliations, minimal components of smooth closed one-form foliations do not have to be open even on a closed manifold:

Example 3.1. *On a 2-torus T^2, there exists a smooth closed one-form ω with a minimal component $\mathcal{C}$ of $\mathcal{F}_\omega$ that is not open. Moreover,*

(i) $\operatorname{int}(\mathcal{C}) = \emptyset$*;*

(ii) *$\mathcal{C}$ is the only minimal component of $\mathcal{F}_\omega$;*

(iii) *$\mathcal{F}_\omega$ has no exceptional leaves;*

(iv) *the union of non-compact compactifiable leaves of $\mathcal{F}_\omega$ is locally dense;*

(v) *$\mathcal{F}_\omega$ has no non-empty minimal open saturated set.*

The construction of the example proceeds as follows. On $M = T^2$, consider a form ω'' defining an irrational winding $\mathcal{F}_{\omega''}$, and a small neighborhood $U' \subset M$ where the form is exact, $\omega''|_{U'} = df''$. We will perturb the form in U' to break down a locally dense leaf into a set of non-compact compactifiable leaves L_i, while preserving other locally dense leaves. This will result in a minimal component in which $\bigcup L_i$ is dense.

First, in a small enough neighborhood U such that $\overline{U} \subset U'$, locally perturb f'' by adding a bulge with a center c_0 and a saddle s_0; see Figure 1 (a). As we will explain below, locally dense leaves of the obtained form ω', $\omega'|_U = df'$, are dense in the complement $\mathcal{C}'$ of the bulge in M, i.e., in the area outside the non-compact compactifiable leaf A (in fact it is a minimal component).

Next, consider a small closed interval $I = I([0,1]) \subset U$ transverse to leaves such that $I(0) = s_0$ and extending outside of the bulge; denote $s_1 = I(1) \in \mathcal{C}'$. Assume the leaf $D \ni s_1$ to be locally dense. The point s_1 divides D in two parts, of which at least one is locally dense; denote this ray by R. Choose a sequence $t_i \in (0,1]$ such that $t_1 = 1$, $t_{i+1} \in (0, \frac{1}{2}t_i]$, and $I(t_{i+1}) = s_{i+1} \in R$. In the own topology of R, the points s_i do not accumulate, because they have no accumulation point on R in M. Therefore, they divide the entire ray R into a sequence of line segments $[s_i, s_{i+1}]$, with $\bigcup [s_i, s_{i+1}]$ being dense in $\mathcal{C}'$.

Finally, in a small neighborhood U_i of each s_i, locally perturb the form to add a bulge with a saddle at s_i and a center c_i; the sequence of bulges of the function f obtained by perturbation of f' in U is shown in Figure 1 (b) seen from the side and in Figure 1 (c) seen from above. By choosing small enough bulges, we can keep the new form ω smooth.

This breaks down the locally dense leaf R into a sequence of compactifiable leaves L_i, with $\bigcup L_i$ being dense in $M \setminus U'$. Let us show that the obtained form ω has the desired properties.

Almost all leaves stayed locally dense after all perturbations. Indeed, consider a leaf L of the original winding ω'' distinct from the leaf broken by s_0 and the leaf D broken by s_1. Consider any two points $x, y \in L \setminus U'$. Since L is linearly connected, there is a segment $[x, y] \subset L$. Since this segment is compact, there exists a neighborhood $V(s_0)$ such that $[x,y] \cap V(s_0) = \emptyset$. Thus, $[x, y]$ is perturbed only by a finite number of bulges of f, and, since the derivative of a curve along the compact segment $[x, y]$ is bounded, $[x, y]$ crosses the perturbation area of each bulge a finite number of times. Non-singular levels of f are connected, as shown in the left-hand part of Figure 1 (c), and coincide with the levels of f'' in $U' \setminus U$. Therefore, each time the segment $[x, y]$ crosses the perturbation area of a bulge following the leaf, it is perturbed but stays connected. Thus the points x, y belong to the same leaf of the perturbed form ω. In particular, x belongs to a locally dense leaf of ω.

We obtained that locally dense leaves of $\mathcal{F}_\omega$ are dense in $M \setminus U'$; in particular, $\mathcal{F}_\omega$ has a minimal component $\mathcal{C}$. Let us show that $\overline{\mathcal{C}}$ covers the whole area outside all bulges, including the corresponding part of U'. Indeed, Theorem 5.1 implies that $\mathcal{C}$ is unique, thus $\mathcal{C}$ is dense in $M \setminus \overline{U'}$, i.e., $M \setminus \overline{U'} \subseteq \operatorname{int}(\overline{\mathcal{C}})$. By Proposition 2.7, the boundary $\partial \operatorname{int}(\overline{\mathcal{C}}) \subset \overline{U'}$ is saturated. However, the only suitable saturated set in $\overline{U'}$ is $\partial \overline{\bigcup_{i=0}^{\infty} \mathcal{C}_i^{max}}$, where $\mathcal{C}_i^{max}$ is the maximal component (connected component of the union of all compact leaves) around the center c_i of the corresponding bulge. By Lemma 3.2, $\operatorname{int}(\mathcal{C}) = \emptyset$ since the compactifiable leaves L_i are dense in $M \setminus U' \subset \overline{\mathcal{C}}$.

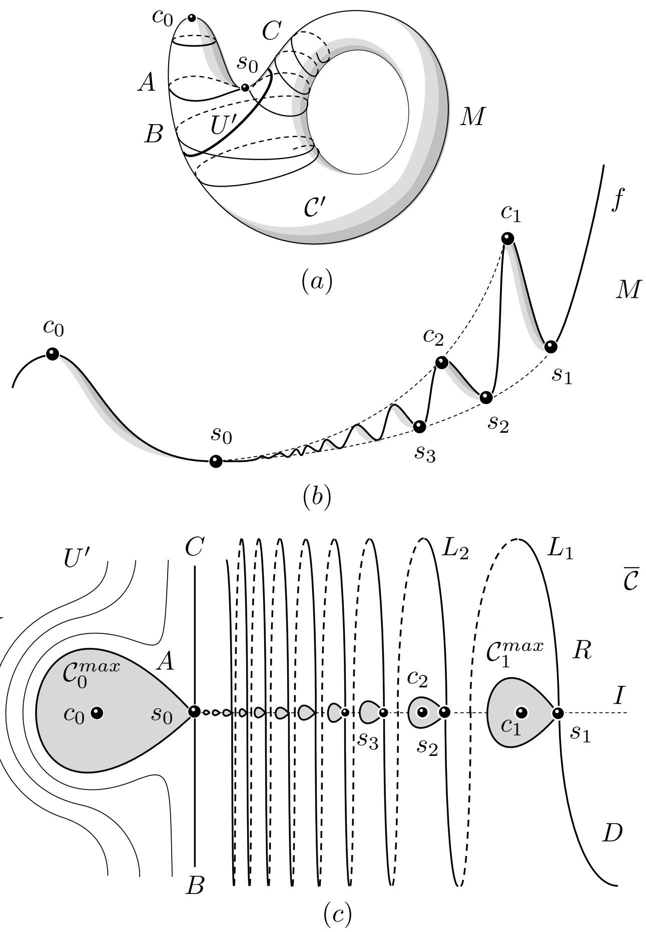

Figure 1: A smooth closed form with a minimal component $\mathcal{C}$ such that $\operatorname{int}(\mathcal{C}) = \emptyset$; see Example 3.1. (*a*) A 2-torus $M = T^2$ with the form ω', which is an irrational winding ω'' locally perturbed by a bulge in a neighborhood U where $\omega' = df'$; $\operatorname{Sing}\omega' = \{ c_0, s_0 \}$; $\mathcal{C}'$ is the only minimal component, which covers the area outside the loop leaf A, and the locally dense leaves $B, C \subset \mathcal{C}'$ are adjacent to the saddle s_0. (*b*) Side view of the neighborhood U, with the height function f resulted from perturbing f' by countably many smaller bulges with saddles s_i. (*c*) The same neighborhood U shown from above, with the maximal components $\mathcal{C}_i^{max}$ corresponding to the bulges shown shaded. The only minimal component $\mathcal{C}$ is dense outside the bulges. The locally dense leaf D is broken down by the singularities s_i into a countably many compactifiable leaves L_i; dashed lines show parts of L_i winding somewhere around T^2 outside of U—“at the other side” of this picture if drawn on the torus.

We obtained that the foliation constructed above has a unique minimal component $\mathcal{C}$, which has empty interior and is dense outside the bulges:

$$\overline{\mathcal{C}} = M \setminus \bigcup_{i=0}^{\infty} (\mathcal{C}_i^{max} \cup \{ c_i \}).$$

Finally, since the union of compact leaves is dense in $\bigcup \overline{\mathcal{C}_i^{max}}$ and the union of compactifiable leaves L_i is dense in its complement, any non-empty open $\mathcal{F}_\omega$-saturated set S includes a compactifiable (possibly compact) leaf L; thus S is not minimal since $S \setminus \overline{L}$ is still non-empty, saturated, and open.

4. Extended minimal components

For the proof of our main result about the number of minimal components, we will introduce objects that are in one-to-one correspondence with minimal components but are easier to count; then we will study their properties.

4.1. Definition of extended minimal components

To compensate for the lack of openness, we enclose each minimal component $\mathcal{C}$ of $\mathcal{F}$ in a non-empty open $\mathcal{F}$-saturated set—an *extended minimal component*:

$$\widehat{\mathcal{C}} = \operatorname{int}(\overline{\mathcal{C}}) \subset M.$$

By Lemma 3.1, $\mathcal{C} \subseteq \widehat{\mathcal{C}}$. Note that even for Morse forms, whose minimal components are open, extended minimal components can contain non-compact compactifiable leaves and even singularities; see Figure 2 and 3.

There is one-to-one correspondence between $\mathcal{C}_i$ and $\widehat{\mathcal{C}}_i$:

Lemma 4.1. *The following conditions are equivalent:* (i) $\widehat{\mathcal{C}}_1 \cap \widehat{\mathcal{C}}_2 \neq \emptyset$*;* (ii) $\mathcal{C}_1 = \mathcal{C}_2$*;* (iii) $\widehat{\mathcal{C}}_1 = \widehat{\mathcal{C}}_2$*. In particular,* $|\{ \widehat{\mathcal{C}}_i \}| = |\{ \mathcal{C}_i \}|$.

Proof. (i) $\Rightarrow$ (ii) Let $\widehat{\mathcal{C}}_1 \cap \widehat{\mathcal{C}}_2 = U \neq \emptyset$. Consider leaves $L_i \subset \mathcal{C}_i$, $i = 1, 2$; by Lemma 3.1, we have $\widehat{\mathcal{C}}_i = \operatorname{int}(\overline{L}_i)$. Thus $U = \operatorname{int}(\overline{L}_1) \cap \operatorname{int}(\overline{L}_2)$, and $U \subseteq \overline{L}_i$. Since U is open, we have $L_i \cap U \neq \emptyset$. Moreover, since U is saturated, $L_i \subset U$, and so $\overline{L}_i \subseteq \overline{U}$. We obtain $\overline{L}_1 = \overline{L}_2 = \overline{U}$; Proposition 3.2 gives $\mathcal{C}_1 = \mathcal{C}_2$. □

Lemma 4.2. *For the foliation of an integrable form* ω*, it holds that* $\omega|_{\partial\widehat{\mathcal{C}}} = 0$.

Proof. Suppose there is $x \in \partial\widehat{\mathcal{C}}$ and $\xi \in T_x(\partial\widehat{\mathcal{C}})$ such that $\omega(\xi) \neq 0$. Since $x \notin \operatorname{Sing}\omega$, in a small neighborhood $U = U(x)$ there exists a curve $\tau \subset U \cap \partial\widehat{\mathcal{C}}$ transversal to leaves, so that $\tau \cap L \neq \emptyset$ for any leaf L such that $L \cap U \neq \emptyset$. Since $\tau \subset \partial\widehat{\mathcal{C}}$ and, by Proposition 2.7, $\partial\widehat{\mathcal{C}}$ is saturated, we obtain $U \subset \partial\widehat{\mathcal{C}}$. However, the boundary of an open set has empty interior; a contradiction. □

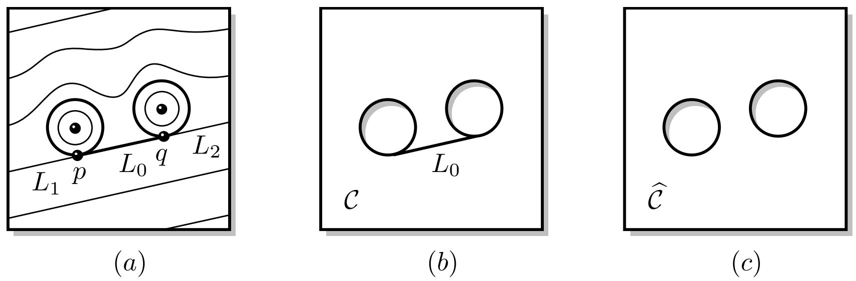

Figure 2: A Morse form ω with an extended minimal $\widehat{\mathcal{C}}$ containing a compactifiable leaf L_0. (a) The form ω on a torus T^2 (represented as a square with identified opposite sides), obtained from an irrational winding by a local perturbation that adds two centers and two saddles p, q located at the same level. The perturbation breaks some locally dense leaf into two locally dense leaves L_1, L_2 and a compactifiable leaf L_0. (b) The only minimal component $\mathcal{C}$ of ω is T^2 without the two closed disks and the compactifiable leaf L_0. (c) The extended minimal component $\widehat{\mathcal{C}}$ includes the compactifiable leaf L_0.

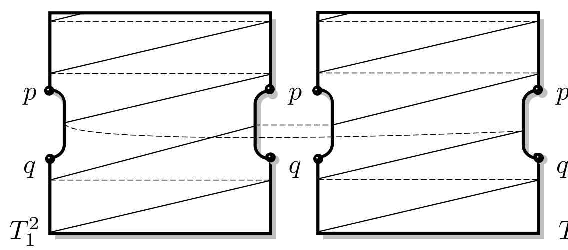

Figure 3: A Morse form ω with $\widehat{\mathcal{C}} \cap \operatorname{Sing}\omega \neq \emptyset$. A double torus $M = T_1^2 \# T_2^2$ with a Morse form ω obtained by taking two tori T_i^2 with an irrational winding, cutting each T_i^2 along a short line transversal to leaves, and gluing the two tori together by the opposite sides of the cut, so that the leaves are glued as shown by the dashed lines. The set $\operatorname{Sing}\omega$ consists of two saddles p, q surrounded by its only minimal component $\mathcal{C} = M \setminus \{p, q\}$, with $\widehat{\mathcal{C}} = M$. Figure adapted from [24], where this type of singularities was studied.

4.2. Extended minimal components of a closed one-form foliation

The number of (extended) minimal components of a smooth closed one-form is finite:

Proposition 4.1. *Let ω be a smooth closed one-form on M. Then*

- (i) *Each extended minimal component $\widehat{\mathcal{C}}$ contains at least two cycles z, z' with incommensurable periods and thus homologically independent in M;*
- (ii) *Such cycles z_i, z_i' from different $\widehat{\mathcal{C}}_i$ are all homologically independent in M;*
- (iii) *$\mathcal{F}_\omega$ has a finite number of (extended) minimal components, which we denote by $m(\omega)$; namely, $m(\omega) \leq \frac{1}{2} b_1(M)$.*

Proof. (i) Suppose all periods of ω in $\widehat{\mathcal{C}}$ are commensurable.

Levitt [25, §0.B] noted that on an arbitrary manifold, all leaves of the foliation of a smooth closed one-form ω with $\operatorname{rk}\omega \leq 1$ are compactifiable. However, we will give an explicit proof for our case.

Consider the subgroup $H \subseteq H_1(M)$ of cycles induced from $\widehat{\mathcal{C}}$, with a basis $\{ z_1, \dots, z_k \}$. By the assumption, $\operatorname{rk}_{\mathbb{Q}}\{\int_{z_1} \omega, \dots, \int_{z_k} \omega\} \le 1$, so there exists $0 \ne \lambda \in \mathbb{R}$ such that $\lambda \int_{z_i} \omega \in \mathbb{Z}$ for all i. Consider a smooth function $F : \widehat{\mathcal{C}} \to S^1$,

$$F(x) = e^{2\pi i \lambda \int_{x_0}^{x} \omega}$$

where $x_0 \in M$. In $\widehat{\mathcal{C}} \setminus \operatorname{Sing} \omega$, the leaves of $\mathcal{F}_\omega$ are connected components of the inverse images $F^{-1}(c)$ and thus are closed (in this set).

This contradicts the fact that $\mathcal{C} \subset \widehat{\mathcal{C}}$ and thus $\widehat{\mathcal{C}}$ contains a locally dense leaf.

(ii) Consider $\widehat{\mathcal{C}}_1, \dots, \widehat{\mathcal{C}}_N$ and the cycles $z_i, z_i' \in H_1(M)$ from (i), i.e., induced from $\widehat{\mathcal{C}}_i$ such that $\operatorname{rk}_{\mathbb{Q}}\{\int_{z_i} \omega, \int_{z_i'} \omega\} = 2$. Consider a combination

$$\sum_i (n_i z_i + n_i' z_i') = 0.$$

For each $k = 1, \dots, N$, it can be rewritten as

$$z = n_k z_k + n_k' z_k' = -\sum_{i \ne k} (n_i z_i + n_i' z_i').$$

The cycle z is induced from both $\widehat{\mathcal{C}}_k$ and $M \setminus \widehat{\mathcal{C}}_k$, thus by the Mayer-Vietoris sequence, it is induced from $\partial \widehat{\mathcal{C}}_k$. Then Lemma 4.2 implies $\int_z \omega = 0 = n_k \int_{z_k} \omega + n_k' \int_{z_k'} \omega$. Since these periods are incommensurable, we obtain $n_k = n_k' = 0$.

(iii). Since the cycles z_i, z_i' from different $\widehat{\mathcal{C}}_i$ are independent, there is at most $b_1(M)$ of them; thus $m(\omega) \le \frac{1}{2} b_1(M)$. By Lemma 4.1, $m(\omega)$ is also the number of minimal components $\mathcal{C}_i$. □

Proposition 4.2. *For any $\widehat{\mathcal{C}}$, there exist $z \in H_1(\widehat{\mathcal{C}})$ and $u \in H_{n-1}(\widehat{\mathcal{C}})$ such that their intersection $z \cdot u \ne 0$.*

PROOF. Denote $U = \widehat{\mathcal{C}}$ and consider homomorphism $\varphi_* : H_1(U) \to H_1(\overline{U})$ induced by the inclusion $\varphi : U \hookrightarrow \overline{U}$. Consider the exact sequence of pairs:

$$\dots \to H_1(\partial \overline{U}) \xrightarrow{i_*} H_1(\overline{U}) \xrightarrow{j_*} H_1(\overline{U}, \partial \overline{U}) \to \dots$$

By Proposition 4.1, there is a closed curve $c \subset U$ such that $\int_c \omega \ne 0$; denote $z = [c]$, then $\varphi_* z \in H_1(\overline{U})$. Lemma 4.2 implies that $k \varphi_* z \notin \operatorname{im} i_*$ for any $k \in \mathbb{Z}$, and thus $k j_* \varphi_* z \ne 0$ for any k.

Since $U = \overline{U} \setminus \partial \overline{U} \subset M$ is open and thus is a smooth manifold, we have the Poincaré-Lefschetz duality

$$D : H_1(\overline{U}, \partial \overline{U}) \to H^{n-1}(U, \mathbb{Z})$$

defined by the cap-product. Denote $\alpha = D j_* \varphi_* z$; by construction, $\alpha \ne 0$. Since $\alpha \in H^{n-1}(U, \mathbb{Z})$ is of infinite order, it can be viewed as an element of $\operatorname{Hom}(H_{n-1}(U), \mathbb{Z})$. Thus there exists a cycle $u \in H_{n-1}(U)$ such that $\alpha(u) \ne 0$; by construction, $z \cdot u \ne 0$. □

5. Main theorem

Recall that $m(\omega)$ is the number of minimal components of the foliation $\mathcal{F}_\omega$, $c(\omega)$ is the number of its homologically independent compact leaves, $b_1'(M)$ is the co-rank of the fundamental group $\pi_1(M)$, and $b_1(M)$ is the first Betti number.

Theorem 5.1. *Let M be a connected smooth closed orientable manifold,* $\dim M \geq 2$*, and ω a smooth closed one-form on it. Then*

(i) $m(\omega) + c(\omega) \leq b_1'(M)$*;*

(ii) $2m(\omega) + c(\omega) \leq b_1(M)$*.*

Proof. Consider a maximal system $\{L_j\}$ of homologically independent compact leaves of $\mathcal{F}_\omega$; obviously, their number $c = c(\omega)$ is finite: $c \leq b_1(M)$. By Proposition 4.1, the number m of extended minimal components $\widehat{\mathcal{C}}_i$ is also finite; by Lemma 4.1, $m = m(\omega)$.

(i) By Proposition 4.2, for each i there exist $z_i \in H_1(M)$ and $u_i \in H_{n-1}(M)$ induced from $\widehat{\mathcal{C}}_i$ such that $z_i \cdot u_i \neq 0$. Let $N_i \subset \widehat{\mathcal{C}}_i$ be submanifolds realizing the cycles u_i.

The system of submanifolds $\{L_i, N_j\}$ is homologically independent: indeed, let

$$\sum_{i=1}^{m} p_i u_i + \sum_{j=1}^{c} q_j [L_j] = 0$$

for some $p_i, q_j \in \mathbb{Z}$. Since $\widehat{\mathcal{C}}_k \cap L_j = \emptyset$ for all j, k, we have $z_k \cdot [L_j] = 0$; similarly, since $\widehat{\mathcal{C}}_k \cap \widehat{\mathcal{C}}_i = \emptyset$ for all $i \neq k$, we have $z_k \cdot u_i = 0$. Thus, intersecting the above equality with each z_k gives $p_k = 0$. Then all $q_j = 0$ since $[L_j]$ are independent.

However, [26, Theorem 2.1] states that the maximum number of homologically independent closed codimention-one submanifolds of M is $b_1'(M)$.

(ii) By Proposition 4.1, each $\widehat{\mathcal{C}}_i$ contains two cycles z_i', z_i'' with incommensurable periods. Consider a combination

$$\sum_{i=1}^{m} (n_i' z_i' + n_i'' z_i'') + \sum_{j=1}^{c} m_j z_j = 0,$$

where $z_1, \dots, z_c$ are 1-cycles dual to the leaves $L_1, \dots, L_c$, i.e., $z_i \cdot [L_j] = \delta_{ij}$. Similarly, intersecting this equality with $[L_k]$ gives $m_k = 0$ for all k. Then Proposition 4.1 (ii) gives all $n_i' = n_i'' = 0$. We obtained that all cycles z_i', z_i'', z_j are independent in $H_1(M)$, so their total number is bounded by $b_1(M)$. □

Which bound in Theorem 5.1 is stronger depends on the structure of the fundamental group $\pi_1(M)$. Denote $b' = b_1'(M)$ and $b = b_1(M)$, then unless $b' = b = 0$ we have:

- if $b' \leq \frac{1}{2} b$, then (i) is stronger (this includes the case $\dim M = 2$);
- if $\frac{1}{2} b < b' < b$, then they are independent;

– if $b' = b$, then (ii) is stronger.

Example 5.1. *For n-torus T^n, the bound (i) is stronger: $b_1(T^n) = n$, whereas $b'_1(T^n) = 1$ (Example 2.1). For $M = \#_{i=1}^p (S^2 \times S^1)$, $p \geq 2$, the bound (ii) is stronger: $b'_1(M) = b_1(M) = p$.*

For each $n \geq 2$, even in the class of Morse forms on smooth closed orientable n-manifolds there are no relations between $m(\omega)$, $c(\omega)$, $b'_1(M)$, and $b_1(M)$ other than those given by (i), (ii), and Theorem 2.4, namely,

$$n = 2: \quad b_1(M) = 2b'_1(M);$$
$$n \geq 3: \quad b'_1(M) = b_1(M) = 0 \text{ or } 1 \leq b'_1(M) \leq b_1(M).$$

Indeed, [17, Theorem 5.2] states that for any $n \geq 2$ and any combination of non-negative m, c, b', and b that satisfies the restrictions specified in this theorem, there exists a connected smooth closed orientable manifold M with $b'_1(M) = b'$ and $b_1(M) = b$ and a Morse form on it with $m(\omega) = m$ and $c(\omega) = c$.

For a given connected smooth closed orientable manifold M, the bound (i) is exact and all intermediate values of $m(\omega) + c(\omega)$ are realized even in the class of Morse forms with compactifiable foliations: [10, Theorem 8] states that for any non-negative $c \leq b'_1(M)$, on M there exists a Morse form with $m(\omega) = 0$ and $c(\omega) = c$.

For a given surface $M = M_g^2$ of genus g, even in the class of Morse forms there are no relations between $m(\omega)$, $c(\omega)$ other than (i), which implies (ii): [10, Proposition 7] states that for any non-negative m and c such that $m + c \leq g$, on the given M there exists a Morse form with $m(\omega) = m$ and $c(\omega) = c$.

We leave open the question of whether on a given manifold M, $\dim M \geq 3$, all pairs of $m(\omega)$ and $c(\omega)$ that satisfy (i) and (ii) can be realized.

6. Condition for compactifiability

For a Morse form foliation, if there exist $b'_1(M)$ homologically independent compact leaves, then all leaves are compactifiable (Corollary 2.1).

By Theorem 5.1, for a smooth closed one-form ω, the condition $c(\omega) = b'_1(M)$ implies that $\mathcal{F}_\omega$ has no locally dense leaves. In fact, it cannot have exceptional and proper non-closed (in $M \setminus \operatorname{Sing} \omega$) leaves either, i.e., all leaves of such foliation are compactifiable:

Theorem 6.1. *Let ω be a smooth closed one-form on M with $c(\omega) = b'_1(M)$. Then:*

(i) *$\mathcal{F}_\omega$ is compactifiable;*

(ii) $\operatorname{rk} \omega \leq b'_1(M)$*;*

(iii) *for any $k = 1, \dots, b'_1(M)$, in any neighborhood of ω, there exists a smooth closed one-form ω' with $\operatorname{rk} \omega' = k$ defining the same foliation, $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$. In particular, $\mathcal{F}_\omega$ can be defined by a rational form, $\operatorname{rk} \omega' \leq 1$.*

PROOF. For $b_1'(M) = 0$ the statements trivially hold, so assume $b_1'(M) \geq 1$.

Recall that $F(\Omega)$ is the space of smooth closed one-forms representing a cohomology class $\Omega \in H^1(M, \mathbb{R})$. By Proposition 2.5, there exists a neighborhood $\mathcal{U}(\omega) \subseteq F([\omega])$ such that, for any $\omega' \in \mathcal{U}(\omega)$, it holds that $H_\omega \subseteq H_{\omega'}$; in particular, $c(\omega') \geq c(\omega) = b_1'(M)$, whereas Theorem 5.1 implies $c(\omega') \leq b_1'(M)$. Thus $c(\omega') = c(\omega) = b_1'(M)$ and $H_{\omega'} = H_\omega$.

By Proposition 2.6, we can choose ω' to be a Morse form, for which $\mathcal{F}_{\omega'}$ is compactifiable by Corollary 2.1. By Proposition 2.3, we have $H_{\omega'}^\ddagger \subseteq \ker[\omega']$. Since $H_{\omega'} = H_\omega$ and $[\omega'] = [\omega]$, we obtain $H_\omega^\ddagger \subseteq \ker[\omega]$. Then Proposition 2.1 gives the desired facts. □

Example 6.1. *Since $b_1'(T^n) = 1$ (Example 2.1), if a smooth closed one-form foliation on a torus T^n has a homologically non-trivial compact leaf, then the foliation is compactifiable and* $\operatorname{rk}\omega \leq 1$.

For $k = 0$, the statement (iii) of Theorem 6.1 may not hold even for a non-singular form, such as $\omega = dx^1$ on T^n, whose foliation cannot be defined by an exact form. For Morse forms, whether a foliation $\mathcal{F}_\omega$ can be defined by an exact form depends on the structure of the so-called directed foliation graph; see [27, Proposition 4.8].

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