The number of split points of a Morse form and the structure of its foliation

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Abstract. Sharp bounds are given that connect split points—conic singularities of a special type—of a Morse form with the global structure of its foliation.

1. Introduction and statement of main results

Consider a smooth closed oriented connected n-dimensional manifold M and a smooth closed differential 1-form ω on it, $d\omega = 0$. By the Poincaré lemma, it is locally the differential of a function: $\omega = df$. We also assume f to be a Morse function; then ω is called a *Morse form*.

Morse functions are smooth functions with non-degenerate singularities. Their set is open and dense in the space of smooth functions [11], i.e., they are "typical" smooth functions. Likewise, Morse forms are "typical" closed 1-forms: their set is open and dense in the space of all closed 1-forms on M.

The set of singularities $\operatorname{Sing}\omega = \{x \in M \mid \omega_x = 0\}$ of a Morse form is finite. On $M \setminus \operatorname{Sing}\omega$ the form ω defines a *foliation* $\mathcal{F}_\omega$ constructed as follows: For any $x \in M \setminus \operatorname{Sing}\omega$, the equation $\{\omega_x(\xi) = 0\}$ defines a distribution of the tangent bundle T_xM. Since ω is closed, this distribution is integrable; its (connected) integral surfaces are leaves of $\mathcal{F}_\omega$. A leaf $\gamma \in \mathcal{F}_\omega$ *adjoins* a singularity $s \in \operatorname{Sing}\omega$ if $\gamma \cup s$ is connected.

If s has no adjoining leaves (the leaves surrounding it are spheres) then it is called a *center*; we denote the set of all centers by $\Omega_0(\omega)$. If there is exactly one leaf adjoining s then we call s a *transformation point*. If more than one leaf adjoins s (up to four if $\dim M = 2$ and two otherwise) then we call s a *split point*; we denote the set of all split points by $\Omega_1^{sp}(\omega)$.

The motivation behind the terms is that when passing a split point, a leaf splits into two, as in Figure 1 imagining the leaf moving upward; see also Figure 3 (a). In contrast, when passing a transformation point, the leaf keeps its integrity but transforms its shape, as in Figure 3 (b). Our notion of split points coincides with what Levitt [16] referred to as *blocking singularities* because they

1991 *Mathematics Subject Classification.* Primary 57R30; Secondary 58K65.
Key words and phrases. Morse form, singularities, foliation, foliation graph.

Cite this paper as: *Mathematica Slovaca*, 63(2):331–348, 2013.
Pre-print version. Final version: http://dx.doi.org/10.2478/s12175-013-0101-x

are obstacles for continuation of the local holonomy map. However, we believe that the term "split point" better reflects their simple geometrical meaning: they split one leaf into two.

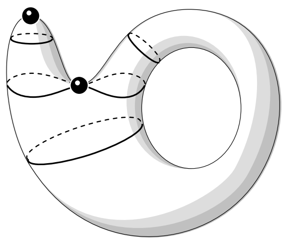


FIGURE 1. A center and a split point.

If $\dim M \geq 3$ then in any non-zero cohomology class there exists a form with only transformation points [15, 20]. Transformation points were thoroughly studied in [1, 15, 16]. We show, however, that it is split points that define the global foliation structure.

Specifically, in this paper we shall study the value

$$d(\omega) = \frac{|\Omega_1^{sp}(\omega)| - |\Omega_0(\omega)|}{2} + 1,$$

which we show to be non-negative. Generally there are almost no other restrictions on this value: in any suitably defined class it takes all integer and half-integer values greater than the minimum for the class (Proposition 5.1). However, we give lower and upper bound on $d(\omega)$ for some important classes of forms and connect this value with the global structure of the foliation.

The intuition behind the value $|\Omega_1^{sp}(\omega)| - |\Omega_0(\omega)|$ is that one can locally add any number of center-and-split-point pairs to a foliation without changing its important properties; see Figure 1. Though not every center is attached to the foliation by a split point—see Figure 3 (*b*), but cf. [2, 3, 20]—we show that the value $d(\omega)$ is still meaningful.

On a 2-dimensional genus g surface M_g^2, all non-center singularities are split points; the Euler characteristic gives $d(\omega) = g$. On the contrary, if $\dim M \geq 3$ then on a given manifold there exist forms with different values $d(\omega)$.

Our main result demonstrates that while $d(\omega)$ is connected with the properties of a finite number of leaves, it defines the global structure of the foliation: the number $c(\omega)$ of homologically independent compact leaves and the number $m(\omega)$ of minimal components of the foliation. These important characteristics of the foliation have been studied in [1, 14, 17]; various bounds on $c(\omega) + m(\omega)$ have been given in [6, 7, 18]. We show (Theorem 4.1) that

$$c(\omega) + m(\omega) \leq d(\omega); \tag{1.1}$$

what is more, for a "typical" Morse form (from a set open and dense in each cohomology class) the inequality turns into equality:

$$c(\omega) + m(\omega) = d(\omega).$$

For a foliation without minimal components, (1.1) implies an exact bound

$$\operatorname{rk}\omega \le d(\omega), \tag{1.2}$$

where $\operatorname{rk}\omega$ is the number of independent (over $\mathbb{Q}$) periods of ω; all integer and half-integer values greater than this bound are reached on M (Proposition 5.4). This can be rephrased as a condition for existence of minimal components: If $|\Omega_1| - |\Omega_0| < 2\operatorname{rk}\omega - 2$, then the foliation has a minimal component (Corollary 5.5).

Since $\Omega_1^{sp}(\omega) \subseteq \Omega_1(\omega)$ (the set of all conic singularities), our results imply lower bounds on $|\Omega_1(\omega)| - |\Omega_0(\omega)|$, which is studied in the Novikov theory of closed 1-forms and their singularities [4, 21].

A Morse form is called *generic* if any its leaf adjoins at most one singularity; such forms are "typical" Morse forms: in each cohomology class their set is open and dense [4]. For generic forms, $d(\omega)$ is integer (Proposition 5.6). While on a given manifold M, a generic form can have an arbitrary large number of conic transformation points (Remark 5.2), the number of its split points (up to $|\Omega_0(\omega)|$) is bounded (Proposition 5.6):

$$0 \le d(\omega) \le b_1'(M), \tag{1.3}$$

where $b_1'(M)$ is the non-commutative Betti number—the maximal rank of a free factor group of $\pi_1 M$ [16]. All intermediate values are reached on M; in particular, the bounds are exact.

If a generic form has no minimal components, (1.2) and (1.3) combine into

$$\operatorname{rk}\omega \le d(\omega) \le b_1'(M),$$

which is also exact with all intermediate values reached (Corollary 5.7).

The paper is organized as follows. In Section 2 we give necessary definitions and prove some auxiliary lemmas. In Section 3 we introduce the notion of split points and describe some their properties. In Section 4 we prove our main result, showing that the number of split points defines the topology of the foliation. Finally, in Section 5 we show that generally there are almost no restrictions on $d(\omega)$, and give exact inequalities for some important special classes of forms in terms of $\operatorname{rk}\omega$ and $b_1'(M)$.

2. Morse form foliation

Let us introduce, for future reference, some useful notions and facts about Morse forms and their foliations.

2.1. **Singularities.** A closed 1-form on M is called a *Morse form* if it is locally the differential of a Morse function. Let ω be a Morse form and $\operatorname{Sing}\omega = \{s \in M \mid \omega(s) = 0\}$ the set of its singularities; this set is finite since the singularities are isolated and M is compact.

Since in a neighborhood of a singularity s we have $\omega = df$, the foliation is defined by a Morse function f; by the Morse lemma there are local coordinates $x_1, \dots, x_n$ such that $x_i(s) = 0$ and $f(x) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$. The number k is called the *index* of the singularity s. In a neighborhood of a singularity of index k and $n - k$ the foliation defined by the levels of f has the same topological structure; we denote the set of such singularities by $\Omega_k(\omega)$, $k \le \frac{n}{2}$.

Singularities $s \in \Omega_0(\omega)$ are called *centers*; a neighborhood of a center foliates into concentric spheres. If ω is exact then $\Omega_0(\omega) \neq \emptyset$ [19]; otherwise in each cohomology class there exists a Morse form without centers: $\Omega_0(\omega) = \emptyset$ [20, Theorem 8.1].

Singularities $s \in \Omega_1(\omega)$ are called *conic*. In a neighborhood of a conic singularity the singular level γ of the corresponding Morse function is (locally) a cone with $\gamma \setminus s$ being not connected. Non-singular levels near s are one-sheeted and two-sheeted hyperboloids; see Figure 5.

At $s \in \Omega_k(\omega)$, $k \ge 2$, the set $\gamma \setminus s$ is connected; nearby non-singular levels are one-sheeted hyperboloids.

2.2. **Foliation.** On $M \setminus \operatorname{Sing}\omega$ the form ω defines a foliation $\mathcal{F}_\omega$. On the whole M we can define a *singular foliation* (which coincides with $\mathcal{F}_\omega$ on $M \setminus \operatorname{Sing}\omega$) as a decomposition of M into leaves; two points $p, q \in M$ belong to the same leaf if there exists a path $\alpha : [0, 1] \to M$ with $\alpha(0) = p$, $\alpha(1) = q$ and $\omega(\dot\alpha(t)) = 0$ for all t. A *singular leaf* contains a singularity. A leaf $\gamma \in \mathcal{F}_\omega$ *adjoins* a singularity s if $\gamma \cup s$ is connected, i.e., if $s \in \overline{\gamma}$ and they belong to the same singular leaf.

A Morse form is called *generic* if each $\gamma \in \mathcal{F}_\omega$ adjoins at most one singularity, i.e., each singular leaf contains a unique singularity. A "typical" Morse form is generic: in each cohomology class on a given M the set of generic forms is open and dense [4].

A leaf $\gamma \in \mathcal{F}_\omega$ is called *compactifiable* if $\gamma \cup \operatorname{Sing}\omega$ is compact; otherwise it is called *non-compactifiable*. If a foliation contains only compactifiable leaves it is called *compactifiable*.

Note that compact leaves are compactifiable. There exists an open neighborhood of a compact leaf γ consisting solely of compact leaves: indeed, integrating ω gives f with $df = \omega$ near γ. Hence, the set covered by all compact leaves is open.

The number of non-compact compactifiable leaves γ_k^0 is finite.

Lemma 2.1 ([10]). *Let $\gamma^0 \in \mathcal{F}_\omega$ be non-compact compactifiable leaf and $\gamma^0 \cup s$ be compact for some $s \in \operatorname{Sing}\omega$. Then there exists a compact leaf $\gamma \in \mathcal{F}_\omega$ which is close to γ^0.*

A *maximal component* $\mathcal{C}_i^{max}$ of the foliation is a connected component of the union of all compact leaves. Unless $\operatorname{Sing}\omega = \emptyset$, each maximal component is a cylinder over a compact leaf:

$$\mathcal{C}_i^{max} \cong \gamma_i \times (0, 1),$$

where the diffeomorphism maps γ_i to leaves of $\mathcal{F}_\omega$. The number of maximal components is finite and can be estimated in terms of homological characteristics of M and the number of singularities of ω [7].

A *minimal component* $\mathcal{C}_j^{min}$ is a connected component of the the set covered by all non-compactifiable leaves. This set is open; it has a finite number $m(\omega)$ of connected components; each non-compactifiable leaf is dense in its minimal component [1]. We say that a minimal component $\mathcal{C}$ *contains* a singularity $s \in \operatorname{Sing}\omega$ if its punctured neighborhood[1] $U'(s) \subseteq \mathcal{C}$.

Components $\mathcal{C}^{max}$ and $\mathcal{C}^{min}$ are open; their boundaries lie in the union $\bigcup \gamma_k^0 \cup \operatorname{Sing}\omega$ of compactifiable leaves and singularities.

The mentioned sets are mutually disjoint and form a partition of M [5]:

$$M = \left(\bigcup \mathcal{C}_i^{max}\right) \cup \left(\bigcup \mathcal{C}_j^{min}\right) \cup \left(\bigcup \gamma_k^0\right) \cup \operatorname{Sing}\omega. \tag{2.1}$$

We call *compact singular quasi-leaf* a connected component Υ of the union of non-compact compactifiable leaves and singularities, i.e., of the set $M \setminus \left(\bigcup \mathcal{C}^{max} \cup \bigcup \mathcal{C}^{min}\right)$; so $\Upsilon = \bigcup \gamma_i^0 \cup \bigcup s_j$, $s_j \in \operatorname{Sing}\omega$. It can be a compact singular leaf or a part of non-compactifiable singular leaf.

2.3. **Foliation graph.** The configuration formed by maximal components in the decomposition (2.1) is described by the *foliation graph.* Rewrite (2.1) as

$$M = \left(\bigcup \mathcal{C}_i^{max}\right) \cup \left(\bigcup P_j\right),$$

where P_j is a connected component of the union P of all non-compact leaves and singularities.

Since $\partial\mathcal{C}_i^{max} \subseteq P$ consists of one or two connected components, each $\mathcal{C}_i^{max}$ adjoins one or two of P_j. This allows representing M as a connected graph Γ (loops and multiple edges are allowed) whose edges are $\mathcal{C}_i^{max}$ and vertices P_j; an edge $\mathcal{C}_i^{max}$ is incident to a vertex P_j if $\partial\mathcal{C}_i^{max} \cap P_j \neq \emptyset$; see Figure 2.

We distinguish between two types of vertices: I-vertices, which do not contain minimal components (they consists solely of compactifiable leaves and singularities) and II-vertices, which in addition contain minimal components.

[1] A punctured neighborhood $U'(s) = U(s) \setminus s$, where $U(s)$ is a neighborhood.

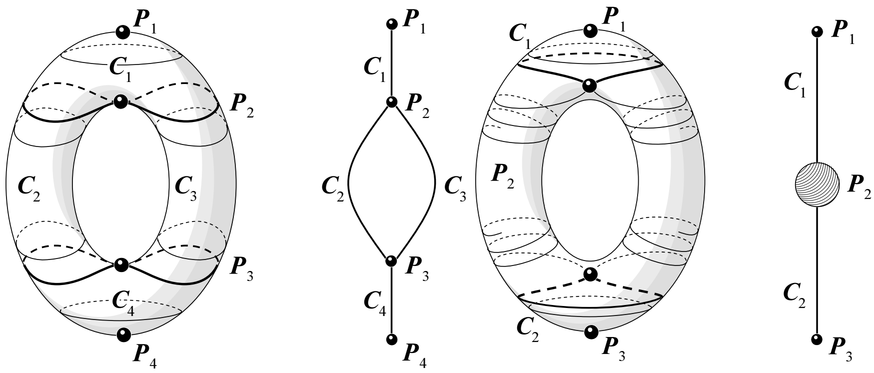


FIGURE 2. Decompositions of the manifold and the corresponding foliation graphs. In the graph on the right, P_2 is a II-vertex; all the other vertices are I-vertices.

The *degree* of a vertex P in the graph is the number of edges incident to this vertex. Geometrically, $\deg P$ is the number of maximal components glued to P. If P is a I-vertex, then it is a compact singular leaf unless $\deg P = 1$, in which case P is a center singularity.

If ω is generic, the vertices of the foliation graph have a rather simple structure:

Lemma 2.2. *Let ω be generic. Then*

(i) *each I-vertex has degree no greater than 3;*
(ii) *each II-vertex contains a unique minimal component.*

Proof. (i) In a small neighborhood of a compact singular leaf P the form is exact, so the leaves of the foliation are levels of a Morse function. Since P contains a unique singularity, close levels can have one or two connected components, which are leaves. So $\deg P \leq 3$.

(ii) Consider a connected component ∂ of $\partial\mathcal{C}^{min}$, which is a compactfiable leaf compactified by one singularity. By Lemma 2.1, there exists a compact leaf close to $\partial\mathcal{C}^{min}$. Thus what is attached to $\mathcal{C}^{min}$ by ∂ is an edge. □

2.4. **Graph-theoretic facts.** Let Γ be a connected graph with V vertices P_i and E edges. The following simple facts can be found, e.g., in [12].

The degree sum formula states that

$$2E = \sum \deg P_i. \tag{2.2}$$

The *cycle rank* $m(\Gamma)$ of the graph is the number of its independent cycles;

$$m(\Gamma) = E - V + k, \tag{2.3}$$

where k is the number of connected components of Γ. In particular, $E \geq V - k$.

If the graph Γ is considered a 1-dimensional simplicial complex, then

$$m(\Gamma) = b_1(\Gamma), \tag{2.4}$$

the first Betti number.

3. Split points

We call a non-center singularity a *split point* if more than one leaf adjoins it; otherwise it is a *transformation point*. We denote the set of split points by $\Omega_1^{sp}(\omega)$. Obviously, only conic singularities can be split points, $\Omega_1^{sp}(\omega) \subseteq \Omega_1(\omega)$, because for a singular leaf γ at any other singularity s the set $\gamma \setminus s$ is connected.

At a split point, the two parts of the cone (without the singularity) globally lie in different leaves. When passing such a singularity, one leaf splits up into two; see Figure 3 (a). At a conic transformation point, the two parts of the cone happen to globally lie in the same leaf, so that when passing such a singularity the leaf only changes its homotopy type; see Figure 3 (b).

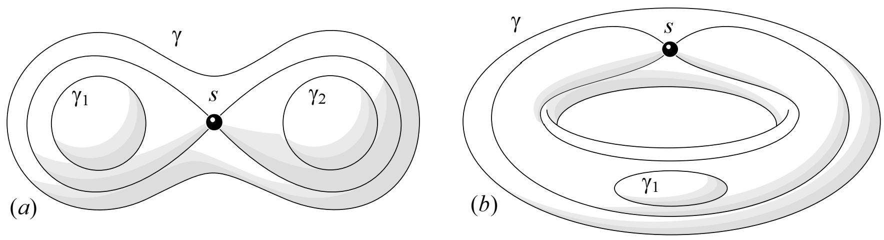


FIGURE 3. (a) s is a split point: the leaf γ splits on it up into γ_1 and γ_2; (b) s is a transformation point: the leaf γ transforms on it into γ_1.

The number of split points defines the structure of the foliation graph. If $|\Omega_1^{sp}(\omega)| = 0$, then the foliation graph is either (a) a chain or circle without II-vertices ($\mathcal{F}_\omega$ is compactifiable) or (b) a unique II-vertex ($\mathcal{F}_\omega$ is minimal).

The following two statements are useful for the proof of our main theorem. Recall that a compact singular quasi-leaf Υ is a connected component of $M \setminus \left(\bigcup \mathcal{C}^{max} \cup \bigcup \mathcal{C}^{min}\right)$; $\Upsilon = \bigcup \gamma_i^0 \cup \bigcup s_j$, $s_j \in \operatorname{Sing} \omega$. Denote by $S^\Upsilon \subseteq \Omega_1^{sp}(\omega) \cap \Upsilon$ the set of split points that adjoin only leaves in Υ; this excludes from $\operatorname{Sing} \omega \cap \Upsilon$ all transformation points and split points adjoining a non-compactifiable leaf.

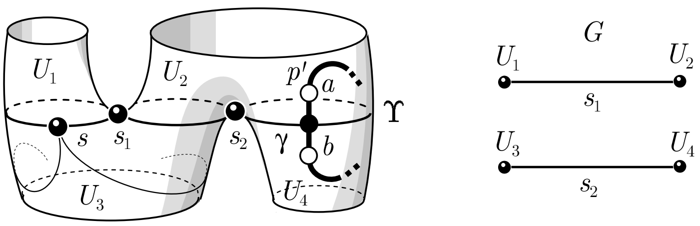
Definition 3.1 ([22]). A *regular neighborhood* U of $X \subset M$ in M is a locally flat, compact submanifold of M, which is a topological neighborhood of X such that the inclusion $X \hookrightarrow U$ is a simple homotopy equivalence, and X is a strong deformation retract of U.

Since a quasi-leaf is a subcomplex of M, it has a regular neighborhood [13].

Lemma 3.2. *Let* $\dim M \geq 3$ *and* Υ *be a compact singular quasi-leaf. Denote by* $d(\Upsilon)$ *the number of connected components of* $U \setminus \Upsilon$*, where* U *is a regular neighborhood of* Υ*. Then*

$$|S^{\Upsilon}| \geq d(\Upsilon) - 2. \tag{3.1}$$

Proof. Denote by U_i connected components of $U \setminus \Upsilon$; then $d(\Upsilon) = |\{U_i\}|$; see Figure 4 (`



(a) (b)

FIGURE 4. (a) The regular neighborhood U of a compact singular quasi-leaf Υ; $S^{\Upsilon} = \{s_1, s_2\}$; $s \notin S^{\Upsilon}$ attaches a minimal component to Υ. A closed path $p' \subset U$ with $[p'] \cdot \Upsilon \neq 0$ is impossible. (b) Graph G is not connected.

We can assume that near $s \in S^{\Upsilon}$ the boundary ∂U forms a one-sheeted and a two-sheeted hyperboloids, see Figure 5. Consider a graph[2] $G = \{\{U_i\}, S^{\Upsilon}\}$, where two vertices U_i, U_j are connected by a conic singularity $s \in S^{\Upsilon}$ if locally they correspond to the opposite sheets of the two-sheeted hyperboloid; see Figure 4 (b). We will show that G is not connected; then (3.1) follows from (2.3).

Consider an equivalence relation R on $U \setminus \Upsilon$: two points a, b are equivalent if they are connected by a path $p \subset U$ such that $p(t) \in U \setminus \Upsilon$ far from S^{Υ}, and p is allowed to cross Υ near $s \in S^{\Upsilon}$ as shown in Figure 5 to connect sheets of the two-sheeted hyperboloid.

Since U is a submanifold of M and U_i is open in U, each U_i is path-connected and thus all points in U_i are equivalent under R. Thus R induces an equivalence relation on the graph G; its equivalence classes are exactly connected components of G. It remains to show that R has more than one equivalence class.

Consider two close points $a, b \in U$ lying at the opposite sides of a leaf $\gamma \subseteq \Upsilon$; see Figure 4. Suppose they are equivalent under R, i.e., are connected by a path p. For the closed curve $p' = p \cup [a, b]$, the intersection index $[p'] \cdot \Upsilon$ is odd and thus nonzero, which contradicts the fact that Υ is a strong deformation retract of

[2]Loops and multiple edges are allowed.

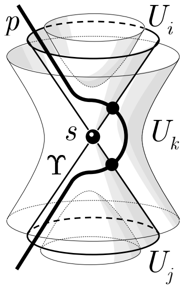


FIGURE 5. U and p' near $s \in S^\Upsilon$.

U. Thus a and b are not R-equivalent, the corresponding U_i belong to different connected components of G, and (2.3) gives (3.1). □

Proposition 3.3. *Let* $\dim M \geq 3$. *If a vertex* P *of the foliation graph* Γ *contains* m *minimal components, then*

$$|P \cap \Omega_1^{sp}(\omega)| \geq \deg P + 2m - 2. \tag{3.2}$$

Proof. Consider a graph Γ' whose vertices are (minimal or maximal) components of $\mathcal{F}_\omega$ and compact singular quasi-leaves Υ_i, and edges are connected components U_{ij} of $U_i \setminus \Upsilon_i$ for a small regular neighborhood U_i of Υ_i. This is a bipartite graph: an edge can only connect a quasi-leaf Υ with a component $\mathcal{C}$, but not two Υ or two $\mathcal{C}$. The value $d(\Upsilon_i)$ from Lemma 3.2 is the degree of the vertex Υ_i in this graph.

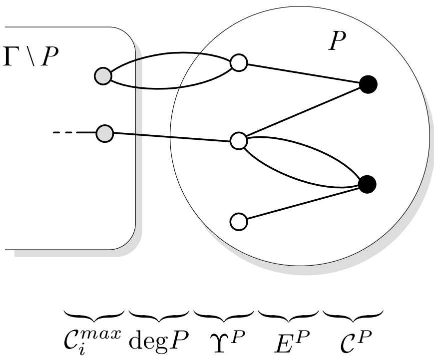


FIGURE 6. A vertex P of the foliation graph Γ as graph Γ'.

A vertex P of the foliation graph Γ is a maximal connected subgraph of Γ' that does not contain any maximal components $\mathcal{C}_i^{max}$; see Figure 6. Denote by $\Upsilon^P \subseteq \{\Upsilon_i\}$, $\mathcal{C}^P \subseteq \{\mathcal{C}_i^{min}\}$, and $E^P \subseteq \{U_{ij}\}$ the sets of vertices and edges belonging to this subgraph; $|\mathcal{C}^P| = m$. Obviously, $\sum_{\Upsilon\in\Upsilon^P} d(\Upsilon) = \deg P + |E^P|$, thus by Lemma 3.2,

$$|\bigcup_{\Upsilon\in\Upsilon^P} S^\Upsilon| \geq \deg P + |E^P| - 2|\Upsilon^P|. \tag{3.3}$$

Each $U_{ij} \in E^P$ attaches to its Υ a minimal component, thus adding to it at least one split point not from S^Υ, so

$$|\,\Omega_1^{sp}(\omega) \cap P \setminus \bigcup_{\Upsilon\in\Upsilon^P} S^\Upsilon| \geq |E^P|. \tag{3.4}$$

Adding (3.3) to (3.4), we obtain $|\Omega_1^{sp}(\omega)\cap P| \geq \deg P + 2|E^P| - 2|\Upsilon^P|$, where (2.3) applied to the subgraph P gives $|E^P| \geq |\Upsilon^P| + m - 1$. □

4. Main theorem

In the sequel we shall study the properties of the value

$$d(\omega) = \frac{|\Omega_1^{sp}(\omega)| - |\Omega_0(\omega)|}{2} + 1. \tag{4.1}$$

For a two-dimensional genus g surface M_g^2, the Euler characteristic gives $d(\omega) = g$.

Recall that a form is generic if each of its singular leaves contains a unique singularity. A minimal component is called *weakly complete* if it contains[3] no split points [16]. For $\dim M \geq 3$ the set of generic forms with weakly complete minimal components is known to be open and dense in a cohomology class [8], so such forms are "typical" in their class.

Theorem 4.1. *Let M be a smooth closed oriented manifold and ω a Morse form on it. Then*

$$c(\omega) + m(\omega) \leq d(\omega), \tag{4.2}$$

where $c(\omega)$ is the number of homologically independent compact leaves of the foliation $\mathcal{F}_\omega$ and $m(\omega)$ is the number of its minimal components.

For generic forms with weakly complete minimal components it holds

$$c(\omega) + m(\omega) = d(\omega). \tag{4.3}$$

Note that at least for $\dim M \geq 3$ typically the equality holds; in particular, the equation holds for generic forms with compactifiable foliation.

Proof. For $M = M_g^2$ it holds $d(\omega) = g$, so (4.2) follows from $c(\omega) + m(\omega) \leq g$ [7] and (4.3) from $c(\omega) + m(\omega) = g$ for the corresponding class of forms [9]. Assume $\dim M \geq 3$.

[3]In the sense of Section 2.2: $U'(s) \subseteq \mathcal{C}$.

(i) Let the foliation graph Γ have V vertices P_i and E edges. Then by (2.3) and (2.2) we have $2m(\Gamma) = \sum_i \deg P_i - 2V + 2 = \sum_i(\deg P_i - 2) + 2$. For vertices P that consist of center singularities $s \in \Omega_0(\omega)$ it holds $\deg P = 1$, and for all other vertices Proposition 3.3 gives $\deg P_i - 2 \le |P_i \cap \Omega_1^{sp}(\omega)| - 2m_i$, where m_i is the number of minimal components in the vertex P_i. We obtain $2m(\Gamma) \le \sum_i |P_i \cap \Omega_1^{sp}(\omega)| - |\Omega_0(\omega)| - 2m(\omega) + 2$, which together with $m(\Gamma) = c(\omega)$ [7] gives (4.2).

(ii) If ω is generic and its minimal components are weakly complete, then except for $\Omega_0(\omega)$ the inequality in Proposition 3.3 turns into equality, and so do the above inequalities.

Indeed, for a non-center I-vertex P, which is a compact singular leaf, Lemma 2.2 gives $\deg P = 3$ if it contains a (unique) split point and $\deg P = 2$ if it contains a transformation point; this turns (3.2) into equality.

A II-vertex P contains a minimal component $\mathcal{C}^{min}$. Since minimal components of ω are weakly complete, $\mathcal{C}^{min}$ does not contain split points, and since ω is generic, each connected component ∂_i of $\partial\mathcal{C}^{min}$ contains a unique singularity. By Lemma 2.2, this singularity must be a split point, $\mathcal{C}^{min}$ is the only minimal component in P, and $\deg P = |\{\partial_i\}| = |P \cap \Omega_1^{sp}(\omega)|$, which again turns (3.2) into equality. □

The theorem allows us to describe the foliation structure in terms of the number of split points. For example, if $|\Omega_1^{sp}(\omega)| < |\Omega_0(\omega)|$ then $c(\omega) = m(\omega) = 0$ and the foliation is compactifiable with all compact leaves being homologically trivial.

If $|\Omega_0(\omega)| \le |\Omega_1^{sp}(\omega)| \le |\Omega_0(\omega)| + 1$ and the cohomology class $[\omega] \ne 0$, then Theorem 4.1 and Corollary 5.3 give $c(\omega) + m(\omega) = 1$ and the foliation has a very simple structure; depending on $\operatorname{rk}\omega$, we have:

(i) If $\operatorname{rk}\omega = 1$ then $\mathcal{F}_\omega$ is compactifiable with $c(\omega) = 1$, i.e., all leaves are homologically equivalent, though they do not have to be diffeomorphic; $\mathcal{F}_\omega$ is similar to $S^1 \times$ something.
(ii) If $\operatorname{rk}\omega > 1$ then $\mathcal{F}_\omega$ has a unique minimal components and all its compact leaves are homologically trivial; $\mathcal{F}_\omega$ is similar to a minimal foliation.

Furthermore, for $\dim M \ge 3$ foliations with such a simple structure are known to exist in any cohomology class $[\omega]$, namely:

$\operatorname{rk}[\omega] = 1$: There exists a compactifiable foliation with $c(\omega) = 1$.

$\operatorname{rk}[\omega] > 1$: There exists a minimal and uniquely ergodic foliation [1].

Indeed, in a non-zero cohomology class there exists a Morse form without centers [20]. Among such forms there exists a form with $\Omega_1^{sp}(\omega) = \emptyset$ [1]; as above, Theorem 4.1 and Corollary 5.3 give $c(\omega) + m(\omega) = 1$. The fact for

$\operatorname{rk}\omega = 1$ follows from (i) above. If $\operatorname{rk}\omega > 1$, then by (ii) above the cycle rank of the foliation graph $m(\Gamma) = c(\omega) = 0$, i.e., Γ is a tree with exactly one II-vertex P that contains the minimal component. If $\mathcal{F}_\omega$ had any compact leaves, then Γ would have edges and thus terminal vertices other than P, which would be centers in $\mathcal{F}_\omega$. Thus the foliation is minimal; by [1], it is uniquely ergodic.

5. Bounds on $d(\omega)$

Since for M_g^2 it holds $d(\omega) = g$, in the sequel we shall assume $\dim M \geq 3$ unless otherwise stated. We shall show that in the general case there are no restrictions on $d(\omega)$ besides a very non-restrictive lower bound. However, compactifiable foliations allow a stronger lower bound on $d(\omega)$ and generic forms allow an upper bound. Naturally, generic compactifiable foliations allow both.

5.1. **No upper bound.** Levitt [15] proved that a small local perturbation within the cohomology class can turn all split points into transformation points. Figure 7 shows the converse: a local—though not small—perturbation within the cohomology class can turn all conic transformation points into split points and centers; each destroyed conic transformation point adds $\frac{1}{2}$ to $d(\omega)$.

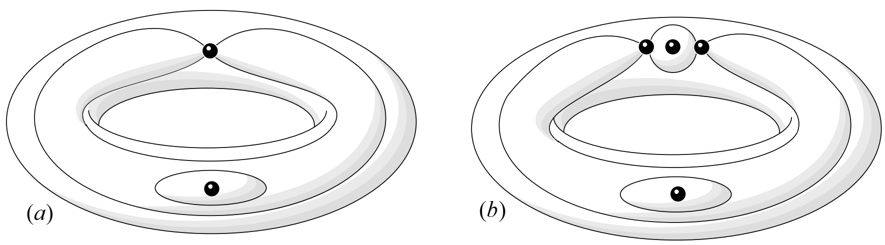


Figure 7. A transformation point can be turned into two split points and one center. Gluing (a) and (b) together by the boundaries into an $S^{n-1} \times S^1$ gives $d(\omega) = \frac{1}{2}$.

In this way an unlimited number of conic transformation points can be added to any foliation:

Proposition 5.1. *In a class of forms with given* $[\omega]$, $c(\omega)$, $m(\omega)$, *and compactifiability,* $d(\omega)$ *takes all integer and half-integer values greater than the minimum for this class.*

Proof. Any foliation $\mathcal{F}_\omega$ can be locally modified preserving all its important characteristics so that $d(\omega')$ be arbitrary large and take all integer and half-integer values $d \geq d(\omega)$.

Indeed, consider a singular leaf shown in Figure 8. Its inside is $S^1 \times D^{n-2}$; let it be foliated as shown in Figure 7 (b). Its outside leaves are spheres. Any number of such solid spheres can be attached through split points to the foliation as shown in Figure 1. Each such sphere adds to $\operatorname{Sing}\omega$ one transformation point of index 2 (which happens to be conic for $\dim M = 3$; see Figure 8), two centers, and three split points (one of them attaches the sphere to the original foliation), which increases $d(\omega)$ by $\frac{1}{2}$.

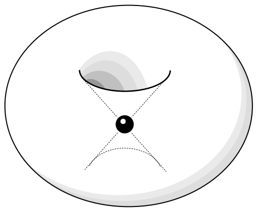


FIGURE 8. A sphere transforms into $S^1 \times S^{n-2}$.

This operation preserves the cohomology class (thus $\operatorname{rk}\omega$), compactifiability, $c(\omega)$, and $m(\omega)$ (as well as some other properties of $\mathcal{F}_\omega$). □

Note that the form constructed in Proposition 5.1 is not generic.

Remark 5.2. In a class of forms with given $d(\omega)$, $[\omega]$, $c(\omega)$, $m(\omega)$, compactifiability, and genericity, the number $|\Omega_1^{tr}(\omega)|$ of conic transformation points takes all values greater than the minimum for this class.

The proof is as in Proposition 5.1, with Figure 7 (a) used instead of Figure 7 (b).

5.2. **Lower bound.** Recall that $[\omega]$ is the cohomology class of ω; $[\omega] = 0$ means globally $\omega = df$.

Corollary 5.3. *It holds*

$$d(\omega) \geq \begin{cases} 0 & \textit{if } [\omega] = 0, \\ 1 & \textit{otherwise.} \end{cases}$$

If $\dim M \geq 3$ *then in each cohomology class the bound is exact and all greater integer and half-integer values are reached.*

Proof. While it is not obvious from the definition (4.1), Theorem 4.1 implies $d(\omega) \geq 0$.

Suppose $d(\omega) \leq \frac{1}{2}$. Theorem 4.1 implies $c(\omega) = m(\omega) = 0$, thus $\mathcal{F}_\omega$ is compactifiable with all compact leaves being homologically trivial. Then $[\omega] = 0$, because for a compactifiable foliation it holds [7]

$$\operatorname{rk}\omega \leq c(\omega). \tag{5.1}$$

Exactness for the case $[\omega] = 0$ follows from the existence of compactifiable foliation with $c(\omega) = \operatorname{rk}\omega$ [6, Theorem 8]; Theorem 4.1 gives $d(\omega) = 0$. On the other hand, in any non-zero cohomology class there exists a Morse form ω without split points and centers: $\Omega_1^{sp}(\omega) = \Omega_0(\omega) = \emptyset$ [15]; $d(\omega) = 1$.

Finally, all greater values are reached by Proposition 5.1. □

For $d(\omega) = 0$ or $\frac{1}{2}$, Corollary 5.3 gives $\omega = df$.

5.3. **Lower bound for compactifiable foliations.** If $\mathcal{F}_\omega$ is compactifiable, then $\operatorname{rk}\omega \le b_1'(M)$, the non-commutative Betti number [15]; indeed, $\operatorname{rk}\omega \le c(\omega)$ [7] and $c(\omega) \le b_1'(M)$ [6].

For $M = M_g^2$ it holds $b_1'(M_g^2) = g$ [6], so for a compactifiable foliation on M_g^2 it holds $\operatorname{rk}\omega \le d(\omega) = g$.

Proposition 5.4. *Let $\dim M \ge 3$ and $\mathcal{F}_\omega$ be compactifiable. Then*

$$\operatorname{rk}\omega \le d(\omega);$$

On a given M this lower bound on $d(\omega)$ is exact and all larger values of $d(\omega)$ are reached. Namely, for any non-negative integer $r \le b_1'(M)$ and any integer or half-integer $d \ge r$, on M there exists a form ω with $\operatorname{rk}\omega = r$ and $d(\omega) = d$.

Proof. By (5.1) and Theorem 4.1 we have $\operatorname{rk}\omega \le c(\omega) \le d(\omega)$, which gives the bound. Let now $r \le b_1'(M)$. There exists a generic form with compactifiable foliation such that $c(\omega) = r$ [6, Theorem 8]. Furthermore, for the same foliation we can choose ω such that $\operatorname{rk}\omega = c(\omega)$ [7, Theorem 4.1]. On the other hand, (4.3) gives $c(\omega) = d(\omega)$.

Finally, all values of $d(\omega) > r$ are reached by Proposition 5.1. □

In particular, a form with compactifiable foliation and a large $\operatorname{rk}\omega$ has many split points—many more than centers. While in any cohomology class with $\operatorname{rk}\omega > 1$ there exist forms without split points, their foliations are minimal [15]; thus the only forms without split points with compactifiable foliation are rational forms—those with $\operatorname{rk}\omega = 1$, i.e., for some $k \in \mathbb{R}$, $k[\omega] \in H^1(M, \mathbb{Z})$.

Given (4.1), Proposition 5.4 can be considered as a condition for existence of minimal components:

Corollary 5.5. *If $|\Omega_1| - |\Omega_0| < 2\operatorname{rk}\omega - 2$, then the foliation has a minimal component: $m(\omega) > 0$.*

5.4. **Upper bound for generic forms.** Recall that a form is called generic if each its singular leaf contains a unique singularity (such forms are "typical"); $b_1'(M)$ is the first non-commutative Betti number: the maximal rank of a free factor group of $\pi_1 M$ [15].

Proposition 5.6. *Let ω be generic. Then $d(\omega)$ is integer and*

$$0 \le d(\omega) \le b_1'(M); \tag{5.2}$$

on a given M the bounds are exact and all integer intermediate values are reached.

Proof. Both the fact that $|\Omega_1^{sp}(\omega)|-|\Omega_0(\omega)|$ is even and the bounds on $|\Omega_1^{sp}(\omega)|-|\Omega_0(\omega)|$ for generic forms were proved in [16].

It was shown in [6, Theorem 8 and Remark 12] that for any c within the bounds (5.2) there exists a generic form ω with $c(\omega)=c$ and $m(\omega)=0$; thus ω is trivially π_1-stable. By Theorem 4.1, $d(\omega)=c(\omega)+m(\omega)=c$. □

Corollary 5.7. *If ω is generic and $\mathcal{F}_\omega$ compactifiable, then*

$$\operatorname{rk}\omega \le d(\omega) \le b_1'(M)$$

and for any non-negative integer r, d such that $r \le d \le b_1'(M)$, on M there exists a generic form ω with $\operatorname{rk}\omega = r$ and $d(\omega)=d$.

Proof. On M there exist generic forms with any $c(\omega)$ between 0 and $b_1'(M)$ [6], and for the same $\mathcal{F}_\omega$ we can choose a form ω' with any $\operatorname{rk}\omega'$ between 0 and $c(\omega)$ [7]. Finally, $d(\omega')=c(\omega')$ by Theorem 4.1. □

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