

Sufficient conditions for the compactifiability of a closed one-form foliation

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ABSTRACT. We study the foliation defined by a closed 1-form on a connected smooth closed orientable manifold. We call such a foliation compactifiable if all its leaves are closed in the complement of the singular set. In this paper, we give sufficient conditions for compactifiability of the foliation in homological terms. We also show that under these conditions, the foliation can be defined by closed 1-forms with the ranks of their groups of periods in a certain range. In addition, we describe the structure of the group generated by the homology classes of all compact leaves of the foliation.

KEYWORDS. Closed one-form foliation, compact leaves, form's rank.

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1. INTRODUCTION

Consider a closed 1-form ω on a connected smooth closed orientable n -dimensional manifold M ; denote by $\text{Sing } \omega$ the set of its singularities. On $M \setminus \text{Sing } \omega$, this form defines a codimension-one foliation \mathcal{F}_ω . Such foliations have important applications in modern physics, for example, in the theory of supergravity [2, 3].

Compact foliations, that is, foliations that consist entirely of leaves closed in M , i.e., compact, are well studied. However, the property of compactness of a foliation is too restrictive: say, manifolds that admit a compact foliation defined by a *Morse form* (locally the differential of a Morse function) are sphere S^n and bundle over S^1 (Proposition 2.5). In addition, compactness is easily destroyed by a local perturbation of the form, for example, by adding a local center—the trivial center-saddle pairing [4].

Instead, we study a weaker but more useful property: compactifiability of the foliation. We call a foliation *compactifiable* if it consists entirely of leaves closed in $M \setminus \text{Sing } \omega$, i.e., such leaves γ that $\gamma \cup \text{Sing } \omega$ is compact. Compactifiable foliations exist on any manifold—say, foliations defined by the levels of a Morse function.

For Morse forms, compactifiability of a foliation has been extensively studied in homological terms related with global characteristics of the manifold [6, 7, 18], as well as using graph-theoretic methods [5, 12, 15]. Using the fact that Morse forms are dense in each cohomology class, in this paper we generalize some of the known facts from Morse forms to arbitrary closed 1-forms.

Specifically, we show that the foliation of a closed 1-form is compactifiable if any of the following conditions holds:

- (i) For any cycle $z \in H_1(M)$ that has zero homological intersection with all compact leaves, $\int_z \omega = 0$ (Theorem 4.2);
- (ii) For some compact leaves $\gamma_1, \dots, \gamma_k \in \mathcal{F}_\omega$, there is no cycle in $H_{n-1}(M)$ homologically independent from them and homologically non-intersecting with them (Theorem 5.3);
- (iii) There exist $h(M)$ homologically independent leaves, where the *isotropy index* $h(M)$ is the maximum number of homologically independent and mutually non-intersecting cycles in $H_{n-1}(M)$ (Corollary 5.4).

The latter simple condition is what best works in practice, because $h(M)$ can be easily computed for many manifolds, such as direct products and connected sums; see (2). It can also be estimated in terms of the structure of the cup product $\smile: H^1(M) \times H^1(M) \rightarrow H^2(M)$ and the Betti numbers $b_1(M)$, $b_2(M)$; see Proposition 2.4. In some cases, this condition allows us to judge on the topology of the whole foliation by a single leaf: for example, if on a torus T^n there is one homologically non-trivial compact leaf, then \mathcal{F}_ω is compactifiable, because $h(T^n) = 1$ (Example 5.5).

Under the same conditions, the form's *rank* (rank of the group of periods) is bounded by $c(\omega)$, the maximum number of homologically independent compact leaves of \mathcal{F}_ω . Moreover, unless all compact leaves of \mathcal{F}_ω are homologically trivial, the same foliation can be defined by closed 1-forms of any rank $1, \dots, c(\omega)$ close to ω (Theorems 4.2, 5.3). Forms that define the same foliation are called *collinear*; they have been studied in [8].

As an important technical result useful to prove basic facts about closed 1-forms, we show that the subgroup generated by the homology

classes of all compact leaves of \mathcal{F}_ω has a basis consisting of homology classes of compact leaves (Theorem 3.1).

The paper is organized as follows. In Section 2, we introduce necessary definitions and facts concerning closed 1-form foliations, isotropic subgroups (subgroups in H_{n-1} with zero homological intersection), isotropy index $h(M)$, Morse forms, and close cohomologous forms. In Section 3, we study the subgroup $H_\omega \subseteq H_{n-1}(M)$ generated by all compact leaves and, specifically, show that it contains a basis consisting of homology classes of compact leaves. In Section 4, we prove a sufficient condition for the compactifiability of a closed 1-form foliation in terms of integrals on cycles that do not intersect compact leaves. Finally, in Section 5 we give our main result: a sufficient condition for the compactifiability of the foliation in terms of maximal isotropic subgroups and the number of homologically independent compact leaves.

2. DEFINITIONS AND USEFUL FACTS

2.1. Foliations of closed 1-forms. Let M be a connected smooth closed orientable n -dimensional manifold and ω a smooth closed 1-form on it with the singular set $\text{Sing } \omega = \{x \in M \mid \omega_x = 0\}$. Obviously, $\text{Sing } \omega$ is closed.

This form defines a codimension-one foliation \mathcal{F}_ω on $M \setminus \text{Sing } \omega$. Indeed, on the set $M \setminus \text{Sing } \omega$ the equation $\{\omega(\xi) = 0\}$, where $\xi \in T_*M$, defines a $(n - 1)$ -dimensional distribution. Since $d\omega = 0$, the distribution is integrable, i.e., it is tangent to the leaves of a foliation; we denote this foliation by \mathcal{F}_ω .

A leaf of a codimension-one foliation is either proper (for example, closed in $M \setminus \text{Sing } \omega$), locally dense (its closure has non-empty interior), or exceptional (its closure is transversally like a Cantor set).

We study leaves $\gamma \in \mathcal{F}_\omega$ closed in $M \setminus \text{Sing } \omega$, i.e., such that $\bar{\gamma} \setminus \gamma \in \text{Sing } \omega$, that is, $\gamma \cup \text{Sing } \omega$ is compact.

Definition 1. *A leaf $\gamma \in \mathcal{F}_\omega$ is called compactifiable if $\gamma \cup \text{Sing } \omega$ is compact; otherwise it is called non-compactifiable. A foliation with all leaves being compactifiable is called compactifiable.*

Under this definition, compact leaves, i.e., the leaves closed in M , are compactifiable, too. A compact leaf has a cylindrical neighborhood consisting of leaves diffeomorphic and homotopically equivalent to it:

Lemma 2.1 ([9, Lemma 3.1]). *Let ω be a closed 1-form and $\gamma \in \mathcal{F}_\omega$ a compact leaf. Then for some neighborhood $U(\gamma)$ there exists a*

diffeomorphism

$$\theta: \gamma \times (-\varepsilon, \varepsilon) \rightarrow U(\gamma)$$

such that $\theta(\gamma, t) = \gamma_t \in \mathcal{F}_\omega$ for any $t \in (-\varepsilon, \varepsilon)$.

In such a neighborhood, we can vary integrals along curves by small perturbation of the form leaving the foliation intact; see a more detailed discussion for Morse forms in [9, Lemma 4.6 ff.]:

Lemma 2.2. *Let ω be a closed 1-form, $\gamma \in \mathcal{F}_\omega$ a compact leaf, and $\alpha: [0, 1] \rightarrow M$ a curve transverse to leaves, $\gamma \cap \alpha = p \in M$. Then for any neighborhood $\mathcal{U}(\omega) \subseteq \Omega^1(M)$ and any neighborhood $U = U(\gamma) \subset M$ there exists $0 < \delta \in \mathbb{R}$ such that for any $c \in (1 - \delta, 1 + \delta)$ there exists a closed 1-form $\omega' \in \mathcal{U}(\omega)$ such that $\omega' = \omega$ on $M \setminus U$, $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$, and $\int_\alpha \omega' = c \int_\alpha \omega$.*

Proof. Consider a small enough cylindrical neighborhood $U_\varepsilon = U_\varepsilon(\gamma) \subset U$ from Lemma 2.1 such that α is transverse to leaves in U_ε . On $(-\varepsilon, \varepsilon)$ choose a positive function being large or small enough near 0 and constant 1 near the ends. It induces on M a function f constant on leaves, $f \equiv 1$ outside U_ε . Since f is constant on leaves, we have $d(f\omega) = df \wedge \omega = 0$, thus the form $\omega' = f\omega$ is closed and $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$, while $\int_\alpha \omega'$ can be varied. \square

The *rank* of the form is $\text{rk } \omega = \text{rk}_{\mathbb{Q}} \text{im}[\omega]$, where $[\omega]: H_1(M) \rightarrow \mathbb{R}$ is the integration map. Obviously, $0 \leq \text{rk } \omega \leq b_1(M)$, the Betti number. A form ω is *exact* if $\text{rk } \omega = 0$.

2.2. Isotropic subgroups. A subgroup $H \subseteq H_{n-1}(M)$ is *isotropic* if it is dual to a subgroup with trivial cup-product, i.e., if it consists of homologically non-intersecting cycles: $z \cdot z' = 0$ for any $z, z' \in H$. Any subgroup generated by one cycle is isotropic. Since leaves do not intersect, the subgroup H_ω generated by the homology classes of all compact leaves of \mathcal{F}_ω is isotropic.

Maximal isotropic subgroups are studied in [11, 19]. For a given manifold M , maximal isotropic subgroups can have different ranks; see Example 2.3. The set $\mathcal{H}(M)$ of ranks of maximal isotropic subgroups can be calculated for various types of manifolds [11]; in particular, for the connected sum and direct product, it holds:

$$(1) \quad \begin{aligned} \mathcal{H}(M \sharp N) &= \mathcal{H}(M) + \mathcal{H}(N), \\ \mathcal{H}(M \times N) &= \{1\} \cup \mathcal{H}(M) \cup \mathcal{H}(N), \end{aligned}$$

except that $\mathcal{H}(M \times N) = \mathcal{H}(M)$ if $b_1(N) = 0$, the Betti number; we denote $A + B = \{a + b \mid a \in A, b \in B\}$. Obviously, $\mathcal{H}(S^1) = \{1\}$; then $\mathcal{H}(T^n) = \{1\}$ and $\mathcal{H}(M_g^2) = \{g\}$ (surface of genus g).

Example 2.3. By (1), $\mathcal{H}(M_2^2 \times S^1) = \{1, 2\}$; see Figure 1: one maximal isotropic subgroup is $\langle [N] \rangle$, with $N = M_2^2$; the other is $\langle [T_1], [T_2] \rangle$, where $T^i = z_i \times S^1$, $z_1, z_2 \subset N$ being homologically independent non-intersecting closed curves.

Isotropy index $h(M) = \max \mathcal{H}(M)$ is the maximum rank of an isotropic subgroup; (1) gives

$$(2) \quad \begin{aligned} h(M \sharp N) &= h(M) + h(N), \\ h(M \times N) &= \max\{h(M), h(N)\}. \end{aligned}$$

For instance, $h(T^n) = 1$, $h(M_g^2) = g$, and $h(M_2^2 \times S^1) = 2$.

The value $h(M)$ can be estimated in terms of the cup product $\smile: H^1(M) \times H^1(M) \rightarrow H^2(M)$ and the Betti numbers $b_i(M)$:

Proposition 2.4 ([11, Proposition 15]). *Let $\dim M \geq 2$. Denote $k = \dim \ker \smile$. Then:*

(i) *It holds*

$$\frac{b_1(M) + k b_2(M)}{b_2(M) + 1} \leq h(M) \leq \frac{b_1(M) b_2(M) + k}{b_2(M) + 1},$$

in particular, if $b_2(M) = 1$, then

$$h(M) = \frac{1}{2}(b_1(M) + k).$$

(ii) *If \smile is surjective, then*

$$h(M) \leq k + \frac{1}{2} + \sqrt{\left(b_1(M) - k - \frac{1}{2}\right)^2 - 2b_2(M)}.$$

In this paper, we give a sufficient condition for the compactifiability of a foliation in terms of $h(M)$ (Corollary 5.4).

2.3. Morse form foliation. A *Morse form* ω is a closed 1-form that is locally the differential of a Morse function. A Morse form foliation is much simpler than that of a general closed 1-form. Its singular set $\text{Sing } \omega$ is finite; its leaves are either compactifiable, i.e., closed in $M \setminus \text{Sing } \omega$, or locally dense ([1, 13]), the number of its non-compact compactifiable leaves being finite. The structure of Morse form foliations is well studied [7, 14].

Compactifiable Morse form foliations exist on any manifold, though compact Morse form foliations exist on a very restricted class of manifolds:

Proposition 2.5. *Let ω be a Morse form. If all leaves of \mathcal{F}_ω are compact, then M is either a sphere or a bundle over S^1 .*

Proof. If $\text{Sing } \omega = \emptyset$, then M is a bundle over S^1 [22]. Let now $\text{Sing } \omega \neq \emptyset$. Near a singularity $p \in \text{Sing } \omega$, ind $p = k$, in a suitable coordinate system x^i the leaves of a Morse form are defined by the equation

$$(3) \quad - \sum_{i=1}^k (x^i)^2 + \sum_{i=k+1}^n (x^i)^2 = \text{const},$$

with $x^i(p) = 0$ for all i . If there are different signs in (3), i.e. $k \neq 0, n$, then a suitable section of a leaf containing the level $(x^i)^2 - (x^j)^2 = 0$ is conic and thus its closure includes the singularity. Therefore a compact \mathcal{F}_ω cannot have singularities other than centers (ind $p = 0, n$). By the Reeb theorem [21], M is homeomorphic to S^n . \square

In this paper, we partially generalize the following statements from Morse forms to arbitrary closed 1-forms.

Denote by $H_\omega \subseteq H_{n-1}(M)$ a subgroup generated by the homology classes of all compact leaves of \mathcal{F}_ω .

Proposition 2.6 ([17, Theorem in Section 2]). *Let ω be a Morse form. If H_ω is a maximal isotropic subgroup, then the foliation \mathcal{F}_ω is compactifiable.*

For a subgroup $H \subseteq H_{n-1}(M)$, denote

$$H^\dagger = \{z \in H_1(M) \mid z \cdot H = 0\},$$

where \cdot is the cycle intersection. Obviously, $A \subseteq B$ implies $B^\dagger \subseteq A^\dagger$.

Proposition 2.7 ([6, Theorem 7]). *Let ω be a Morse form. The foliation \mathcal{F}_ω is compactifiable if and only if $H_\omega^\dagger \subseteq \ker[\omega]$.*

Denote $c(\omega) = \text{rk } H_\omega$. For a Morse form foliation, $c(\omega)$ is the maximum number of homologically independent compact leaves of \mathcal{F}_ω [6, Theorem 4]. In case of compactifiable foliation, this number bounds the rank of a form that can define \mathcal{F}_ω :

Proposition 2.8 ([8, Proposition 4.8]). *Let ω be a Morse form. If \mathcal{F}_ω is compactifiable, then there exists a Morse form ω' such that $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$ and $\text{rk } \omega' = k$ if and only if*

$$\alpha \leq k \leq c(\omega),$$

where $\alpha = 0$ or 1 , depending on the topology of the so-called directed foliation graph of \mathcal{F}_ω .

2.4. Close cohomologous forms. Foliation defined by close forms can have very different topological structure: for example, a form with rational coefficients on a torus defines a compact foliation, while a close form with an irrational coefficient defines a winding, i.e., a minimal foliation.

However, foliations of closed 1-forms that are both cohomologous and close have, in some sense, similar topology. For example, compact leaves are stable under small perturbations of the form in its cohomology class. In particular, denote by $F(\Omega)$ the space of closed 1-forms representing a class $\Omega \in H^1(M, \mathbb{R})$; then:

Proposition 2.9 ([9, Theorem 3.1]). *Let ω be a closed 1-form. There exists a neighborhood $\mathcal{U}(\omega) \subseteq F([\omega])$ such that for any $\omega' \in \mathcal{U}(\omega)$ it holds $H_\omega \subseteq H_{\omega'}$.*

The following statement makes facts concerning Morse form foliations useful in the study of arbitrary closed 1-form foliations:

Proposition 2.10 ([20, Ch. 2, Theorem 1.25]). *Let M be a closed manifold. The set of Morse forms is open and dense in each cohomology class $\Omega \in H^1(M, \mathbb{R})$.*

3. THE SUBGROUP $H_\omega \subseteq H_{n-1}(M)$ GENERATED BY ALL COMPACT LEAVES

Denote by H_ω the group generated by the homology classes of all compact leaves.

A generating set of a free group might not contain its basis, e.g., $\mathbb{Z} = \langle 2, 3 \rangle$; the generators in $\mathbb{Z}^2 = \langle (1, 0), (2, 3), (3, 2) \rangle$ are indivisible. However, any set of homology classes of non-intersecting connected codimension-one submanifolds of M contains a basis of the group it generates (unless the latter is trivial); in particular:

Theorem 3.1. *Let \mathcal{F}_ω be a closed 1-form foliation on M and $H_\omega \subseteq H_{n-1}(M)$ be the subgroup generated by the homology classes of all compact leaves; $H_\omega \neq 0$. Then:*

- (i) *In H_ω , there exists a basis e consisting of homology classes of leaves: $e = \{[\gamma_1], \dots, [\gamma_{c(\omega)}]\}$, $\gamma_i \in \mathcal{F}_\omega$, $c(\omega) = \text{rk } H_\omega$.*
- (ii) *For any compact leaf $\gamma \in \mathcal{F}_\omega$, it holds $[\gamma] = \sum_{i \in I} \pm [\gamma_i]$, where $[\gamma_i] \in e$ and $I \subseteq \{1, \dots, c(\omega)\}$.*

Proof. (i) Since M is closed and oriented, its $H_{n-1}(M)$ is a finitely generated free abelian group, and so is $H_\omega \subseteq H_{n-1}(M)$. Since it is Noetherian, from any set of its generators a finite subset can be chosen: $H_\omega = \langle [\gamma_1], [\gamma_2], \dots, [\gamma_m] \rangle$.

Suppose $[\gamma_i]$ are dependent. Let M' be the result of cutting M open along all γ_i ; then $M' = \bigcup_j M'_j$ has at least two connected components M'_j , $\partial M'$ consisting of two copies γ_i^\pm of each γ_i . Since M is connected, there is $\gamma_k \subset M$ such that γ_k^+ and γ_k^- lie in different components: assume $\gamma_k^+ \in \partial M'_1$. Then $[\gamma_k] = \sum_{i \in I_1} \pm [\gamma_i]$, where the signs depend on the orientation. Thus we can remove γ_k from the generating set and repeat the process until obtaining a basis e of H_ω .

(ii) Consider a compact leaf $\gamma \in \mathcal{F}_\omega$, $[\gamma] \notin e$. Obviously, $\gamma \subset \text{Int}(M')$ and cutting M' along γ will result in two connected components, M'_+ and M'_- , with $\partial M'_+ = \gamma^+ \cup \bigcup \gamma_i^+$; thus $[\gamma] = \sum_{i \in I} \pm [\gamma_i]$. \square

The same fact about Morse form foliations has been proved using the finiteness of the foliation graph [6, Theorem 4].

4. COMPACTIFIABILITY OF FOLIATIONS WITH $H_\omega^\ddagger \subseteq \ker[\omega]$

Here we generalize the sufficient condition from Proposition 2.7, and partially Proposition 2.8, from Morse forms to arbitrary closed 1-forms.

Lemma 4.1. *Let $\tilde{\omega}$ be an exact 1-form on a compact manifold \tilde{M} with boundary. Then $\mathcal{F}_{\tilde{\omega}}$ is compactifiable.*

Proof. Let $\tilde{\omega} = df$. A leaf $\gamma \in \mathcal{F}_{\tilde{\omega}}$ is a connected component of $S = f^{-1}(a) \setminus \text{Crit}(f)$, a level set of f without critical points $\text{Crit}(f) = \text{Sing } \tilde{\omega}$; since S is locally path-connected, its connected components are path-connected. Consider $p \in \bar{\gamma}$. Then $p \in f^{-1}(a)$ and in some spherical neighborhood of p there exist coordinates x^i such that $f^{-1}(a) = \{x^1 = 0\}$; thus $p \in \gamma$. We obtained that γ is closed in $M \setminus \text{Sing } \tilde{\omega}$; in particular, $\gamma \cup \text{Sing } \tilde{\omega}$ is compact. \square

Recall that for an $H \subseteq H_{n-1}(M)$, we denote

$$H^\ddagger = \{z \in H_1(M) \mid z \cdot H = 0\},$$

where \cdot is the cycle intersection. Recall also that $c(\omega) = \text{rk } H_\omega$ is the maximum number of homologically independent compact leaves of \mathcal{F}_ω (Theorem 3.1).

If the integral along any cycle that does not intersect compact leaves is zero, then the foliation is compactifiable:

Theorem 4.2. *Let ω be a closed 1-form on M such that $H_\omega^\ddagger \subseteq \ker[\omega]$, where $[\omega]: H_1(M) \rightarrow \mathbb{R}$ is the integration map. Then:*

- (i) \mathcal{F}_ω is compactifiable;
- (ii) $\text{rk } \omega \leq c(\omega)$ and, if $c(\omega) \geq 1$, for any $k = 1, \dots, c(\omega)$ in any neighborhood of ω there exists a closed 1-form ω' defining the same foliation, $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$, with $\text{rk } \omega' = k$.

Proof. (i) By Theorem 3.1, we have $H_\omega = \langle [\gamma_1], \dots, [\gamma_{c(\omega)}] \rangle$, where $\gamma_i \in \mathcal{F}_\omega$ are homologically independent compact leaves. Let \tilde{M} be the result of cutting M open along γ_i . Then \tilde{M} is connected, and $\partial\tilde{M}$ consists of two copies of each γ_i . Denote by $\varphi: \tilde{M} \rightarrow M$ the gluing map; the induced form $\tilde{\omega} = \varphi^*\omega$ defines on $\tilde{M} \setminus \text{Sing } \tilde{\omega}$ a foliation $\mathcal{F}_{\tilde{\omega}}$ that coincides with \mathcal{F}_ω defined on the whole $M \setminus \text{Sing } \omega$, i.e., $\varphi(\mathcal{F}_{\tilde{\omega}}) = \mathcal{F}_\omega$.

Consider a closed curve $s \subset \tilde{M}$. Obviously, $[\varphi(s)] \cdot H_\omega = 0$, so $[\varphi(s)] \in H_\omega^\dagger \subseteq \ker[\omega]$. We have

$$\int_s \tilde{\omega} = \int_s \varphi^*\omega = \int_{\varphi(s)} \omega = 0,$$

i.e., the form $\tilde{\omega}$ is exact. By Lemma 4.1, the foliation $\mathcal{F}_{\tilde{\omega}}$ is compactifiable, and so is \mathcal{F}_ω .

(ii) Since the form $\tilde{\omega} = df$ is exact on \tilde{M} , the periods of ω are defined by the cycles $D[\gamma_1], \dots, D[\gamma_{c(\omega)}] \in H_1(M)$; thus $\text{rk } \omega \leq c(\omega)$.

Denote by s_i curves realizing $D[\gamma_i]$ and transverse to leaves in mutually non-intersecting neighborhoods $U_i = U(\gamma_i)$. Then $s_i = \alpha_i \cup s'_i$, where $\alpha_i = s_i \cap U_i$; see Figure 2. By Lemma 2.2, all $\int_{\alpha_i} \omega$ can be independently varied by small perturbation of the form, ω' , without changing the foliation, thus varying

$$\text{rk } \omega' = \text{rk}_{\mathbb{Q}} \left\{ \int_{s_i} \omega' \mid i = 1, \dots, c(\omega) \right\}$$

from 1 to $c(\omega)$. □

Whether \mathcal{F}_ω can be defined by an exact form, $\text{rk } \omega' = 0$, depends on other factors; see, e.g., Proposition 2.8.

Corollary 4.3. *Let ω be a closed 1-form on a closed orientable surface M_g^2 of genus g . If \mathcal{F}_ω has g homologically independent compact leaves, then it is compactifiable.*

Proof. We exploit the fact that $\dim \mathcal{F}_\omega = \text{codim } \mathcal{F}_\omega$. Let $\gamma_1, \dots, \gamma_g$ be homologically independent leaves of \mathcal{F}_ω . Then

$$H_1(M_g^2) = \langle [\gamma_1], \dots, [\gamma_g], D[\gamma_1], \dots, D[\gamma_g] \rangle$$

is a basis, where D is a Poincaré duality map, i.e., $D[\gamma_i] \cdot [\gamma_j] = \delta_{ij}$; thus $c(\omega) = g$ and $H_\omega = \langle [\gamma_1], \dots, [\gamma_g] \rangle$. For any $z \in H_\omega^\dagger$ we have

$$0 = z \cdot [\gamma_j] = \left(\sum n_i [\gamma_i] + \sum m_i D[\gamma_i] \right) \cdot [\gamma_j] = m_j$$

for all j ; thus $H_\omega^\dagger \subseteq H_\omega$. Since obviously $H_\omega \subseteq \ker[\omega]$, by Theorem 4.2, the foliation \mathcal{F}_ω is compactifiable. □

For Morse form foliations, this fact has been known [16].

5. COMPACTIFIABILITY OF FOLIATIONS WITH MAXIMAL ISOTROPIC H_ω

As our main result, we generalize Proposition 2.6 from Morse forms to arbitrary closed 1-forms, using the fact that Morse forms are dense in any cohomology class.

Proposition 5.1. *The set of closed 1-forms with a given maximal isotropic H_ω is open in the space $F(\Omega)$ of closed 1-forms representing a class $\Omega \in H^1(M, \mathbb{R})$.*

Proof. Suppose a form $\omega \in F(\Omega)$ defines a foliation with H_ω being maximal isotropic. By Proposition 2.9, there exists a neighborhood $U = U(\omega) \subset F(\Omega)$ such that for any $\omega' \in U$ it holds $H_\omega \subseteq H_{\omega'}$. Since H_ω is maximal isotropic, $H_{\omega'} = H_\omega$ and thus is also maximal isotropic. \square

Proposition 5.2. *Let ω be a closed 1-form such that some compact leaves of \mathcal{F}_ω generate a maximal isotropic subgroup $H \subseteq H_{n-1}(M)$. Then:*

- (i) $H = H_\omega$, the subgroup generated by all compact leaves;
- (ii) $H_\omega^\dagger \subseteq \ker[\omega]$.

Proof. (i) is obvious since H_ω is isotropic. (ii) By Propositions 5.1 and 2.10, in the cohomology class of ω there exists a Morse form ω' with $H_{\omega'} = H_\omega$. By Proposition 2.6, its foliation $\mathcal{F}_{\omega'}$ is compactifiable, and thus, by Proposition 2.7, $H_{\omega'}^\dagger \subseteq \ker[\omega']$, i.e.,

$$H_\omega^\dagger = H_{\omega'}^\dagger \subseteq \ker[\omega'] = \ker[\omega]. \quad \square$$

We obtain an important particular case of Theorem 4.2:

Theorem 5.3. *Let ω be a closed 1-form on M such that the subgroup generated by the homology classes of some compact leaves of its foliation \mathcal{F}_ω is maximal isotropic. Then:*

- (i) \mathcal{F}_ω is compactifiable;
- (ii) $\text{rk } \omega \leq c(\omega)$ and, if $c(\omega) \geq 1$, for any $k = 1, \dots, c(\omega)$ in any neighborhood of ω there exists a closed 1-form ω' defining the same foliation, $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$, with $\text{rk } \omega' = k$.

Whether \mathcal{F}_ω can be defined by an exact form is, again, not discussed here.

Recall that $h(M) = \max \mathcal{H}(M)$ is the maximum rank of an isotropic subgroup of $H_{n-1}(M)$.

Corollary 5.4. *If \mathcal{F}_ω has $h(M)$ homologically independent compact leaves, then it is compactifiable.*

In this case, the statement (ii) above about the form ranks holds, too.

Example 5.5. *If on a torus T^n there exists a homologically non-trivial compact leaf, then the foliation \mathcal{F}_ω is compactifiable and $\text{rk } \omega \leq 1$. Indeed, $h(T^n) = 1$.*

Example 5.6. *If on M_g^2 there exist g homologically independent compact leaves, then the foliation \mathcal{F}_ω is compactifiable. Indeed, $h(M_g^2) = g$. This has also been shown as Corollary 4.3.*

Note that Theorem 4.2 is stronger than Theorem 5.3; in particular, the converse to Theorem 5.3 (i) is not true. The following counterexample gives a compactifiable foliation that satisfies the conditions of Theorem 4.2 ($H_\omega^\ddagger \subseteq \ker[\omega]$) but not Theorem 5.3 (H_ω is not maximal isotropic):

Counterexample 5.7. *Consider a foliation on $M = T^2$ with two centers and two saddles, defined by an exact Morse form as shown in Figure 3. Since the form is exact, $H_\omega^\ddagger \subseteq \ker[\omega] = H_1(M)$; in particular, \mathcal{F}_ω is compactifiable. However, since all its compact leaves are homologically trivial, $H_\omega = 0$ (and $H_\omega^\ddagger = H_1(M)$). Thus H_ω is not maximal isotropic, since any 1-generated subgroup in $H_{n-1}(M)$ is isotropic.*

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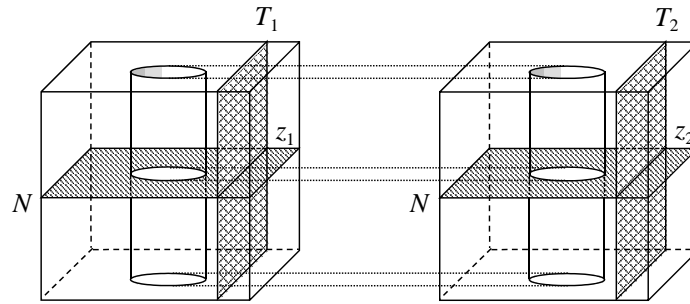


FIGURE 1. Two maximal subgroups of different ranks: $\langle [N] \rangle$ and $\langle [T_1], [T_2] \rangle$. The sides of each cube are identified to form a 3-torus. In each 3-torus, a solid 2-torus represented by the vertical cylinder is removed, and the two obtained boundaries are identified to form one 3-manifold. In this 3-manifold, three 2-submanifolds are shown: N is a double 2-torus represented by the two horizontal squares glued by the central circles, and T_1 and T_2 are 2-tori represented by the vertical squares. Figure borrowed from [11].

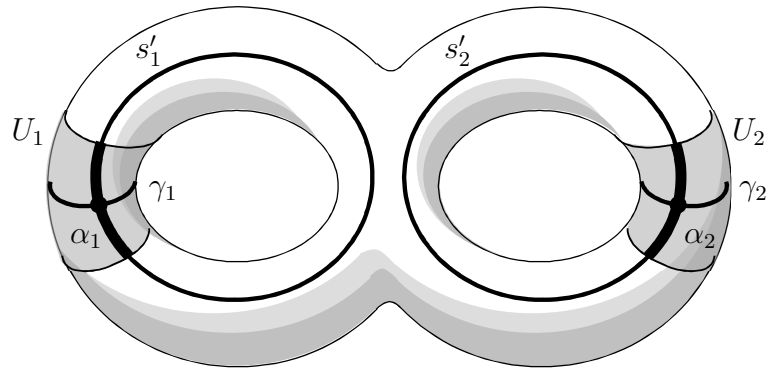


FIGURE 2. By small perturbation of the form in the neighborhoods U_1, U_2 of the compact leaves γ_1, γ_2 without changing the foliation, the integrals along the closed curves $s_i = \alpha_i \cup s'_i$ can be made commensurable or incommensurable.

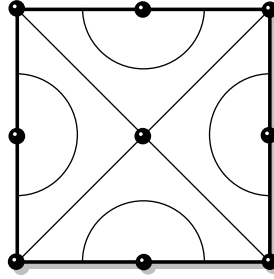


FIGURE 3. Compactifiable Morse form foliation on T^2 with $H_\omega^\pm \subseteq \ker[\omega]$ but not maximal isotropic H_ω . The sides of the square are identified to form a torus. The foliation of the height function on a suitably tilted torus has two centers shown on the sides, two saddles shown in the middle and on the corners, and four non-compact compactifiable leaves shown as a cross; the rest of the the torus is covered by homologically trivial compact leaves. Figure borrowed from [10].